B.Sc. MATHEMATICS

I YEAR - I SEMESTER
COURSE CODE: 7BMA1C1

CORE COURSE - I –CALCULUS

Unit – I

Successive Differentiation – Leibnitz formula – Envelopes – curvatures – circle, radius and centre of curvature – Evolutes.

Unit – II

Polar Coordinates – Radius of curvature in polar coordinates, p-r equation of a curve – Asymptotes – Method of finding asymptotes – problems

Unit – III

Definite Integrals and their properties –problems – Integration by parts — Reduction formulae - Bernoulli's formula.

Unit – IV

Double and triple integrals and their properties – Jacobian – Change of order of integration.

Unit – V

Beta and Gamma functions – properties – problems

Text Book:

1. Calculus, Volume I (edi.2015) andVolume II (edi.2016) by S.Narayanan and T.K.ManicavachagomPillay, S.Viswanathan (Printers and Publishers) Pvt. Ltd.

Unit I	Chapter 3 (Volume I) sections 1 & 2 Chapter 10 up to section 2.5 (Volume I)
Unit II	Chapter 10 sections 2.6, 2.7 (Volume I) Chapter 11 upto section 7
Unit III	Chapter 1 sections 11, 12, 13, 14, 15.1(Volume II)
Unit IV	Chapter 5 sections 1, 2, 3, 4 (Volume II) Chapter 6 sections 1, 2 (Volume II)
Unit V	Chapter 7 sections 2, 3, 4, 5, (Volume II)

Books for Reference:

1. Calculus and Fourier series by Dr. M.K.Venkataraman and Mrs. Manorama Sridhar, The National Publishing Company, Chennai.
2. Calculus Volume I and Volume II by Dr. S.Arumugam and A.Thangapandi Isaac, New Gamma Publishing House, Palayamkottai.

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B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-3

PART-1

R.RAJALAKSHMI [G.L],GACW,RAMANATHAPURAM

Contents

DIFFERENTIATION

- The essence of calculus is the derivative.
  The derivative is the instantaneous rate of change of a function with respect to one of its variables.

- The derivative of a function $y = f(x)$ of a variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x.

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

FUNCTION | **DERIVATIVES**

- $y = (f(x))^n$ — $\frac{dy}{dx} = nf(x)(f(x))^{n-1}$
- $y = e^{f(x)}$ — $\frac{dy}{dx} = f(x)(e)^{f(x)}$
- $y = \log[f(x)]$ — $\frac{dy}{dx} = \frac{f(x)}{f(x)}$
- $y = sin(f(x))$ — $\frac{dy}{dx} = f(x)\cos[f(x)]$
- $y = cos(f(x))$ — $\frac{dy}{dx} = -f(x)\sin[f(x)]$
- $y = tan(f(x))$ — $\frac{dy}{dx} = f(x)\,sec^2[f(x)]$
- $y = cot(f(x))$ — $\frac{dy}{dx} = -f(x)cosec^2[f(x)]$

(1) $\frac{d}{dx}(x) = 1$

(2) $\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = x^n$, $n \neq -1$

(3) $\frac{d}{dx}\ln|x| = \frac{1}{x}$

(4) $\frac{d}{dx}(\sin x) = \cos x$

(5) $\frac{d}{dx}(-\cos x) = \sin x$

(6) $\frac{d}{dx}(\tan x) = \sec^2 x$

(7) $\frac{d}{dx}(-\cot x) = \csc^2 x$

(8) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(9) $\frac{d}{dx}(-\csc x) = \csc x \cot x$

(10) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(11) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(12) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(13) $\frac{d}{dx}(e^x) = e^x$

(14) $\frac{d}{dx}(\frac{a^x}{\ln a}) = a^x$

Trigonometric Identities

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Sum Identities
Addition Formulas

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

Difference Identities
Subtraction Formulas

$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$

Double Angle Formulas

$$\sin 2a = 2\sin a\cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a$$

$$= 2\cos^2 a - 1$$

$$= 1 - 2\sin^2 a$$

$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a}$$

Co-function Identities

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$$

$$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$$

$$\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$$

Even-Odd Identities

$$\sin(-\theta)=-\sin\theta$$

$$\cos(-\theta)=\cos\theta$$

$$\tan(-\theta)=-\tan\theta$$

$$\csc(-\theta)=-\csc\theta$$

$$\sec(-\theta)=\sec\theta$$

$$\cot(-\theta)=-\cot\theta$$

Half-Angle Formulas

$$\sin\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{\sin\theta}$$

$$=\frac{\sin\theta}{1+\cos\theta}$$

$$=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

Sum-to-Product Formulas

$$\sin a+\sin b=2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin a-\sin b=2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\cos a+\cos b=2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos a-\cos b=-2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Product-to-Sum Formulas

$$\sin a\sin b=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\cos a\cos b=\frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$$

$$\sin a\cos b=\frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$

$$\cos a\sin b=\frac{1}{2}\left[\sin(a+b)-\sin(a-b)\right]$$

SUCCESSIVE DIFFERENTIATION

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its successive derivatives be denoted by $f'(x), f''(x), \dots, f^{(n)}(x)$.

Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

⋮

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

Calculation Of n^{th} Derivatives

i. n^{th} Derivative of e^{ax}

Let $y = e^{ax}$

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$\vdots$$
$$y_n = a^n e^{ax}$$

ii. n^{th} Derivative of $(ax + b)^m$, m is a +ve integer greater than n

Let $y = (ax + b)^m$

$$y_1 = m\,a(ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax + b)^{m-2}$$
$$\vdots$$
$$y_n = m(m-1)\ldots(m-n+1)a^n(ax + b)^{m-n} = \frac{m!}{(m-n)!}a^n(ax + b)^{m-n}$$

iii. **n^{th} Derivative of $y = \log(ax+b)$**

Let $y = \log(ax+b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\,a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$$

NOTE: *if* $y = \frac{a}{(ax+b)}$

$$y_1 = \frac{-a^2}{(ax+b)^2}$$

$$y_2 = \frac{2!\,a^3}{(ax+b)^3}$$

.

.

$$y_n = \frac{(-1)^n n!\,a^{n+1}}{(ax+b)^{n+1}}$$

iv. **n^{th} Derivative of $y = \sin(ax+b)$**

Let $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax+b)$

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

v. n^{th} Derivative of $y = e^{ax}\sin(ax+b)$

Let $y = e^{ax}\sin(bx+c)$

$$y_1 = a\,e^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$$

$$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}\,[\,r\,cos\alpha\sin(bx+c) + r\,sin\alpha\cos(bx+c)]$$

Putting $a = r\,cos\alpha,\ b = r\,sin\alpha$

$$= e^{ax}\ r\ \sin(bx+c+\alpha)$$

Similarly $y_2 = e^{ax}\ r^2\ \sin(bx+c+2\alpha)$

$\vdots$

$$y_n = e^{ax}\ r^n\ \sin(bx+c+n\alpha)$$

where $r^2 = a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly if $y = e^{ax}\cos(ax+b)$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha)$$

$$= e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	$\boldsymbol{n^{th}}$ **Derivative**
$y = e^{ax}$	$\boldsymbol{y_n} = a^n e^{ax}$
$y = (ax+b)^m$	$\boldsymbol{y_n} = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & m > 0, m > n \\ 0, \quad m > 0, & m < n, \\ n!\, a^n, & m = n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, & m = -1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$
$y = e^{ax} \sin(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$
$y = e^{ax} \cos(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$

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UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-3[1.1-1.5]

PART-2

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CONTENTS

EXERCISE:13

ASSIGNMENT:

UNIT - I

CALCULUS - 7BMA1C1 ; TEXT BOOK - T.K MANICAVACHAGOM PILLAY, S. NARAYANAN, S. VISWANATHAN

SUCCESSIVE DIFFERENTIATION :

Excercise : 13

1) Find the n^{th} differential coefficient of

(i) $\sin^3 x$

Soln:

$\sin^3 x = \sin^2 x \cdot \sin x$

$= \left[\frac{1-\cos 2x}{2}\right] \sin x$

$= \frac{1}{2}\left[\sin x - \cos 2x \sin x\right]$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$= \frac{1}{2}\left[\sin x - \frac{1}{2}\left[\sin(2x+x) - \sin(2x-x)\right]\right]$

$= \frac{1}{2}\left[\sin x - \frac{1}{2}\left[\sin 3x - \sin x\right]\right]$

UNIT-I

CALCULUS - 7BMA1C1; TEXT BOOK: S. NARAYANAN, S. VISWANATHAN, T.K. MANICKAVASAGAM PILLAY

$$= \frac{1}{2}\left[\frac{3\sin x}{2} - \frac{\sin 3x}{2}\right]$$

$$\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$$

$$D^n \sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$

$$D^n(\sin^3 x) = \frac{1}{4}\left[3(1)^n \sin\left(\frac{n\pi}{2}+x\right) - (3)^n \sin\left(\frac{n\pi}{2}+3x\right)\right]$$

$$= \frac{3}{4}\sin\left(\frac{n\pi}{2}+x\right) - \frac{(3)^n}{4}\sin\left(\frac{n\pi}{2}+3x\right)$$

2) $\cos^4 x$

Soln:

$$\cos^4 x = (\cos^2 x)^2 = \left[\frac{1+\cos 2x}{2}\right]^2$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \cos^2 2x\right]$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right]$$

$$= \frac{1}{2\times 4}\left[2 + 4\cos 2x + 1 + \cos 4x\right]$$

$$= \frac{1}{8}\left[3 + 4\cos 2x + \cos 4x\right]$$

$$\cos^4 x = \frac{3}{8} + \frac{4}{8}\cos 2x + \frac{1}{8}\cos 4x$$

$$D^n(\cos^4 x) = \frac{4}{8}(2)^n \cos\left(\frac{n\pi}{2} + 2x\right) + \frac{1}{8}(4)^n \cos\left(\frac{n\pi}{2} + 4x\right)$$

$$D^n(\cos^4 x) = 2^{n-1}\cos\left(\frac{n\pi}{2} + 2x\right) + \frac{4^n}{8}\cos\left(\frac{n\pi}{2} + 4x\right)$$

$D^n[\cos(ax+b)]$
$= a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

3) $\sin^2 x \cos^3 x$

Soln:

$$(2i\sin x)^2 (2\cos x)^3 = 2^2 i^2 \sin^2 x \cdot 2^3 \cos^3 x$$

$$2^{2+3}(-1)\sin^2 x\cos^3 x = -2^5 \sin^2 x \cos^3 x$$

$$(2i\sin x)^2(2\cos x)^3 = \left(x-\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^3$$

$$-2^5\sin^2 x\cos^3 x = \left[\left(x-\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^2\right]\left(x+\frac{1}{x}\right)$$

$$-2^5\sin^2 x\cos^3 x = \left[x^2-\frac{1}{x^2}\right]\left[x+\frac{1}{x}\right]$$

$$= \left[x^4-2+\frac{1}{x^4}\right]\left[x+\frac{1}{x}\right]$$

$$= \left[x^5+x^3-2x-\frac{2}{x}+\frac{1}{x^3}+\frac{1}{x^5}\right]$$

$$\sin^2 x\cos^3 x = \frac{-1}{2^5}\left[\left(x^5+\frac{1}{x^5}\right)+\left(x^3+\frac{1}{x^3}\right)-2\left(x+\frac{1}{x}\right)\right]$$

$$D^n\left[\sin^2 x\cos^3 x\right] = \frac{-1}{2^5}\left[2\cdot 5^n\cos\left(\frac{n\pi}{2}+5x\right)+2\cdot 3^n\cos\left(\frac{n\pi}{2}+3x\right)-2(1)^n\cos\left(\frac{n\pi}{2}+x\right)\right]$$

$$= -\frac{1}{2^4}\left[5^n\cos\left(\frac{n\pi}{2}+5x\right)+3^n\cos\left(\frac{n\pi}{2}+3x\right)-\cos\left(\frac{n\pi}{2}+x\right)\right]$$

$$(2i\sin\theta)^n = \left(x-\frac{1}{x}\right)^n$$

$$(2\cos\theta)^n = \left(x+\frac{1}{x}\right)^n$$

$$x^n+\frac{1}{x^n} = 2\cos n\theta$$

4) $e^{4x}\sin^2 x$

Soln:

$$e^{4x}\cdot\sin^2 x = e^{4x}\left[\frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{2}\left[e^{4x} - e^{4x}\cos 2x\right]$$

$$D^n\left[e^{4x}\sin^2 x\right] = \frac{1}{2}D^n\left[e^{4x}\right] - \frac{1}{2}\left[e^{4x}\cdot\cos 2x\right]$$

$$\frac{1}{2}D^n\left[e^{4x}\right] = \frac{4^n e^{4x}}{2} = \frac{(2^2)^n}{2}\cdot e^{4x} = 2^{2n-1}e^{4x}$$

$$\frac{1}{2}D^n\left[e^{4x}\cdot\cos 2x\right] = \frac{(20)^{n/2}}{2}e^{4x}\cos(2x + n\tan^{-1}(1/2))$$

$$D^n\left[e^{4x}\cdot\sin^2 x\right] = 2^{2n-1}\cdot e^{4x} - \frac{(20)^{n/2}}{2}\cdot e^{4x}\cos\left(2x + n\tan^{-1}(1/2)\right)$$

$$D^n\left[e^{ax}\right] = a^n e^{ax}$$

$$D^n\left[e^{ax}\cos(bx+c)\right]$$

$$= r^n e^{ax}\cos(bx + c + n\phi)$$

$$r = (a^2+b^2)^{1/2}$$

$$\phi = \tan^{-1}(b/a)$$

$$r = (4^2+2^2)^{1/2}$$

$$r = (16+4)^{1/2}$$

$$r = 20^{1/2}$$

$$\phi = \tan^{-1}\cdot\left(\frac{2}{4}\right)$$

$$\phi = \tan^{-1}(1/2)$$

11) $\frac{1}{4x^2+8x+3}$

Soln:

$$y = \frac{1}{4x^2+8x+3} = \frac{1}{(2x+1)(2x+3)}$$

$$\frac{1}{(2x+1)(2x+3)} = \frac{A}{(2x+1)} + \frac{B}{(2x+3)}$$

$$1 = A(2x+3) + B(2x+1)$$

$x = -1/2 \Rightarrow 1 = A(2) \Rightarrow A = 1/2$

$x = -3/2 \Rightarrow 1 = B(-2) \Rightarrow B = -1/2$

$$\frac{1}{(2x+1)(2x+3)} = \frac{1}{2(2x+1)} - \frac{1}{2(2x+3)} =$$

$$\frac{1}{(2x+1)(2x+3)} = \frac{1}{2}\left[\frac{1}{(2x+1)} - \frac{1}{(2x+3)}\right]$$

$4x^2+8x+3$

$= 4x^2+6x+2x+3$

$= 2x(2x+3)+1(2x+3)$

$= (2x+1)(2x+3)$

$$y = \frac{1}{2}\left[\frac{1}{(2x+1)} - \frac{1}{(2x+3)}\right]$$

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!\, 2^n}{(2x+1)^{+(n+1)}} - \frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!\, 2^n}{2 \cdot 2^{n+1}}\left[\frac{1}{(x+1/2)^{n+1}} - \frac{1}{(x+3/2)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!\, 2^n}{2 \cdot 2^n \cdot 2}\left[\frac{1}{(x+\frac{1}{2})^{n+1}} - \frac{1}{(x+\frac{3}{2})^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{4}\left[\frac{1}{(x+\frac{1}{2})^{n+1}} - \frac{1}{(x+\frac{3}{2})^{n+1}}\right]$$

$$y = \frac{a}{ax+b}$$

$$y_n = (-1)^n n!\, a^n (ax+b)^{-(n+1)}$$

3) (i) If $x = a(t - \sin t)$, $y = a(1 + \cos t)$ find $\frac{d^2y}{dx^2}$ as a function of t.

Soln:

$x = a(t - \sin t)$ $\qquad$ $y = a(1 + \cos t)$

$\frac{dx}{dt} = a(1 - \cos t)$ $\qquad$ $\frac{dy}{dt} = a(-\sin t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$$

$$\frac{dy}{dx} = \frac{2 \sin t/2 \cos t/2}{2 \sin^2 t/2} = \cot(t/2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot t/2)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$\sin t = 2 \sin t/2 \cos t/2$

$1 - \cos t = 2 \sin^2 t/2$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t/2 \cdot \frac{1}{2} \cdot \frac{dt}{dx}$$

$$= -\frac{1}{2\sin^2 t/2} \cdot \frac{1}{a(1-\cos t)}$$

$$= -\frac{1}{2a\sin^2 t/2 \times 2\sin^2 t/2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-1}{4a\sin^4 t/2}}$$

3 (ii) Find $\frac{d^2y}{dx^2}$ if $x = \sqrt{\sin 2t}$ and $y = \sqrt{\cos 2t}$

Soln:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$y = \sqrt{\cos 2t} \qquad\qquad x = \sqrt{\sin 2t}$$

$$\frac{dy}{dt} = \frac{1}{\cancel{2}\sqrt{\cos 2t}} \cdot (-\sin 2t) \cdot \cancel{2} \qquad\qquad \frac{dx}{dt} = \frac{1}{\cancel{2}\sqrt{\sin 2t}} \times \cos 2t \cdot \cancel{2}$$

$$\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{1}{2\sqrt{f(x)}} \times f'(x)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin 2t/\sqrt{\cos 2t}}{\cos 2t/\sqrt{\sin 2t}} = -\frac{\sin 2t\sqrt{\sin 2t}}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = -\left(\frac{\sin 2t}{\cos 2t}\right)^{3/2} = -(\tan 2t)^{3/2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan 2t)^{3/2} = -\frac{3}{\cancel{2}}(\tan 2t)^{1/2} \cdot \sec^2 2t \cdot \cancel{2} \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-3(\sin 2t)^{1/2}(\sin 2t)^{1/2}}{(\cos 2t)^{1/2+2+1}} = \frac{-3\sin 2t}{(\cos 2t)^{7/2}}$$

8) (i) If $y^2 = (x-a)(x-b)$ Show that

$$\frac{d^3}{dx^3}\left[\left(\frac{d^2y}{dx^2}\right)^{-2/3}\right] = 0$$

Soln: $\frac{dy}{dx} = y_1$

$\frac{d^2y}{dx^2} = y_2$

$\therefore$ L.H.S $= \frac{d^3}{dx^3}\left[y_2^{-2/3}\right]$

$$y^2 = \left[(x-a)(x-b)\right] = x^2 - (a+b)x + ab$$

$$2yy_1 = 2x - (a+b)$$

$$y_1 = \frac{2x-(a+b)}{2y} = \frac{1}{2}\left[\frac{2x-(a+b)}{y}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{y(2) - (2x - \overline{a+b})y_1}{y^2}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{2y - (2x - \overline{a+b})\left(\frac{2x - \overline{a+b}}{2y}\right)}{y^2}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{4y^2 - (2x - \overline{a+b})^2}{2y^2}\right]$$

$$= \frac{1}{4}\left[\frac{4y^2 - (2x - \overline{a+b})^2}{y^3}\right]$$

$$y_2^{-2/3} = \frac{1}{4}\left[\frac{4y^2 - (2x - \overline{a+b})^2}{y^3}\right]^{-2/3}$$

$$= \left(\frac{1}{4}\right)^{-2/3} \frac{\left[4y^2 - (2x - \overline{a+b})^2\right]^{-2/3}}{(y^3)^{-2/3}}$$

$$y_2^{-2/3} = \left(\frac{1}{4}\right)^{-2/3} y^2 \left[4y^2 - (2x - \overline{a+b})^2\right]^{-2/3}$$

$$y_2^{-2/3} = \left(\frac{1}{4}\right)^{-2/3} y^2 \left[4\left(x^2-(a+b)x+ab\right) - \left(4x^2+(a+b)^2-4(a+b)x\right) \right]$$

$$= \left(\frac{1}{4}\right)^{-2/3} y^2 \left[4x^2 - 4(a+b)x + 4ab - 4x^2 - (a+b)^2 + 4(a+b)x \right]$$

$$= \left(\frac{1}{4}\right)^{-2/3} y^2 \left[4ab - (a^2+b^2+2ab) \right]$$

$$= \left(\frac{1}{4}\right)^{-2/3} y^2 \left[-a^2 - b^2 + 2ab \right]$$

$$= -\left(\frac{1}{4}\right)^{-2/3} y^2 \left[a^2 + b^2 - 2ab \right]$$

$$= -\left(\frac{1}{4}\right)^{-2/3} (a-b)^2 y^2 = -\left(\frac{1}{4}\right)^{-2/3} (a-b)^2 \left(x^2-(a+b)x+ab\right)$$

$$\frac{d^3}{dx^3}\left[y_2^{-2/3}\right] = -\left(\frac{1}{4}\right)^{-2/3} (a-b)^2 \frac{d^2}{dx^2}\left[\frac{d}{dx}\left(x^2-(a+b)x+ab\right)\right]$$

$$= -\left(\frac{1}{4}\right)^{-2/3} (a-b)^2 \frac{d^2}{dx^2}\left[2x - (a+b)\right]$$

$$= -\left(\frac{1}{4}\right)^2 (a-b)^2 \frac{d}{dx}\left[2\right]$$

$$= -\left(\frac{1}{4}\right)^2 (a-b)^2 (0)$$

$$\frac{d^3}{dx^3}\left[y_2^{-2/3}\right] = 0 .$$

Hence proved.

EXERCISE:9(i) Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\cos\theta,\ a = r\sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\sin\theta)^n} - \frac{1}{r^n(\cos\theta + i\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

HOMEWORK PROBLEMS

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75	EXERCISE:13 Q.NO:1 8,10,15
76	Q.NO:4 Q.NO:7
77	Q.NO:9 (ii),(iii),(iv)

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-3[1.6-2.2]

PART-2

R.RAJALAKSHMI [G.L],GACW,RAMANATHAPURAM

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- TEXT BOOK PAGE NO:83 EXAMPLE NO:1
- TEXT BOOK PAGE NO:84 EXAMPLE NO:3
- TEXT BOOK PAGE NO:85 PROBLEM NO:1(5),1(6)
- TEXT BOOK PAGE NO:85 PROBLEM NO:5
- TEXT BOOK PAGE NO:86 PROBLEM NO:11

ASSIGNMENT:

Examples: If $xy = ae^{x} + be^{-x}$, Prove that

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$$

Soln:

Here, $xy = ae^{x} + be^{-x} \rightarrow (1)$

Now differentiating both sides with respect to x, We've

$$y + x\frac{dy}{dx} = ae^{x} - be^{-x}$$

Differentiating both sides of the equation once again, we get

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = ae^{x} + be^{-x}$$

i.e., $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$ [from (1)] ie, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$

Hence proved.

Excercise: 14 PROBLEM. NO: 3

If $y = e^{-x}\cos x$, Prove that $\frac{d^4y}{dx^4} + 4y = 0$

<u>Soln:</u>

$y = e^{-x}\cos x$

$y_1 = e^{-x}(-\sin x) + \cos x(-e^{-x})$

$y_1 = -e^{-x}(\sin x + \cos x)$

$y_2 = -\left\{ e^{-x}(\cos x - \sin x) + (\sin x + \cos x)(-e^{-x}) \right\}$

$y \ \ = \ \ e^{-x}\cos x + e^{-x}\sin x + e^{-x}\sin x + e^{-x}\cos x$

$y_2 = 2e^{-x}\sin x$

$y_3 = 2[e^{-x}\cos x + \sin x(-e^{-x})]$

$y_3 = 2e^{-x}[\cos x - \sin x]$

$y_4 = 2[e^{-x}(-\sin x - \cos x) + (\cos x - \sin x)(-e^{-x})]$

$= 2[-e^{-x}\sin x - e^{-x}\cos x - e^{-x}\cos x + e^{-x}\sin x]$

$= 2[-2e^{-x}\cos x]$

$= -4e^{-x}\cos x$

$y_4 = -4y$ $\qquad \frac{d^4y}{dx^4} + 4y = 0$

EXCERCISE: 14 PROBLEM: 19

If $y = Ax^{n+1} + Bx^{-n}$ P.T $x^2 \frac{d^2y}{dx^2} = n(n+1)y$

Soln:

$$y = Ax^{n+1} + Bx^{-n}$$

$$y_1 = A(n+1)x^{n+1-1} + B(-n)x^{-n-1}$$

$$y_1 = A(n+1)x^{n} - Bn\,x^{-(n+1)}$$

$$y_2 = A(n+1)n\,x^{n-1} - Bn(-(n+1))x^{-n-1-1}$$

$$= An(n+1)x^{n-1} + Bn(n+1)x^{-n-2}$$

$$y_2 = n(n+1)\left[A x^{n-1}\frac{x^{+2}}{x^{+2}} + B\frac{x^{-n}}{x^{2}}\right]$$

$$= \frac{n(n+1)}{x^2}\left[A x^{n-1+2} + B x^{-n}\right]$$

$$x^2 y_2 = n(n+1)\left[A x^{n+1} + B x^{-n}\right]$$

$$x^2 y_2 = n(n+1)\,y$$

$$x^2\frac{d^2y}{dx^2} = n(n+1)\,y$$

Hence proved.

§ 2.1. Leibnitz formula for the n^{th} derivative of a product.

This formula expresses the n^{th} derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If u and v are functions of x, we have

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

i.e. $D(uv) = vDu + uDv.$

Differentiating again with respect to x, we get

$$D^2(uv) = D(v \cdot Du) + D(u \cdot Dv)$$
$$= vD^2u + 2\,Du \cdot Dv + u \cdot D^2v.$$

Similarly

$$D^3(uv) = v \cdot D^3u + 3D^2u \cdot Dv + 3\,Du \cdot D^2v + u \cdot D^3v.$$

However for this process may be continued, it will be seen that the numerical coefficients follow the same law as that of the Binomial Theorem and the indices of the derivatives correspond to the exponents of the Binomial Theorem. Hence

$$\frac{d^n}{dx^n}(uv) =$$

$$\frac{d^n u}{dx^n} v + n C_1 \frac{d^{n-1} u}{dx^{n-1}} \cdot \frac{dv}{dx} + n C_2 \frac{d^{n-2} u}{dx^{n-2}} \cdot \frac{d^2 v}{dx^2}$$

$$+ \ldots + n C_r \cdot \frac{d^{n-r} u}{dx^{n-r}} \cdot \frac{d^r v}{dx^r} + \ldots + n C_1 \frac{du}{dx} \cdot \frac{d^{n-1} v}{dx^{n-1}} + u \cdot \frac{d^n v}{dx^n}$$

§ 2.2. A complete formal proof by induction may be given as follows :

Assume the theorem to be true for some one value of n, i.e., suppose

$$D^n (uv) = u_n v + n C_1 u_{n-1} v_1 + n C_2 u_{n-2} v_2$$
$$+ \ldots + n C_{r-1} u_{n-r+1} v_{r-1} + n C_r u_{n-r} v_r + \ldots + uv_n.$$

Differentiating again we get

$$D^{n+1}(uv) = (u_{n+1}v + u_n v_1) + n C_1 (u_n v_1 + u_{n-1} v_2)$$
$$+ n C_2 (u_{n-1} v_2 + u_{n-2} v_3) + \ldots n C_{r-1} (u_{n-r+2} v_{r-1} + u_{n-r+1} v_r)$$
$$+ n C_r (u_{n-r+1} v_r + u_{n-r} v_{r+1}) + \ldots + \ldots + (u_n v_n + u v_{n+1})$$
$$= u_{n+1} v + (1 + n C_1) u_n v_1 + (n C_1 + n C_2) u_{n-1} v_2$$
$$+ \ldots + (n C_{r-1} + n C_r) u_{n-r+1} v_r + \ldots + u v_{n+1}$$

Now $n C_{r-1} + n C_r = (n+1) C_r$ and so

$$1 + n C_1 = (n+1) C_1$$
$$n C_1 + n C_2 = (n+1) C_2$$
$$n C_2 + n C_3 = (n+1) C_3 \text{ and so on.}$$

$$\therefore \quad D^{n+1}(uv) = u_{n+1} v + (n+1) C_1 u_n v_1 + \ldots$$
$$(n+1) C_r u_{n+1-r} v_r + \ldots + u v_{n+1}$$

Hence if the theorem be true for any value of n, it must be true for the next higher value $n + 1$. It has been seen that it is true for $n = 1$ and therefore it is true for $n = 2$ and therefore for $n = 3$ and so on for all values of n.

This theorem is particularly useful when one of the factors is a small integral multiple of x ; if this be taken as v in the proceding formula, its differential coefficients and the series will consist of only a few terms.

PAGE NO: 83 EXAMPLE. 1

Find the n^{th} differential coefficient of $x^2 \log x$

Soln:

$$\frac{d^n}{dx^n}(uv) = \frac{d^n}{dx^n}(u)v + nC_1 \frac{d^{n-1}}{dx^{n-1}}(u)\frac{dv}{dx} + nC_2 \frac{d^{n-2}}{dx^{n-2}}(u)\frac{d^2v}{dx^2} + \dots + nC_r \frac{d^{n-r}}{dx^{n-r}}(u) \frac{d}{d}$$

$$+ \dots nC_1 \frac{du}{dx} \cdot \frac{d^{n-1}v}{dx^{n-1}} + u \cdot \frac{d^n v}{dx^n}$$

Let $V = x^2$ & $u = \log x$

$$\frac{d^n(x^2 \log x)}{dx^n} = \frac{d^n}{dx^n}(\log x)(x^2) + nC_1 \frac{d^{n-1}(\log x)}{dx^{n-1}} \frac{d}{dx}(x^2) +$$

The successive derivatives of x^2 after the second derivative vanish.

$$D^n(x^2\log x) = \frac{(-1)^{n-1}(n-1)!}{x^n}\,x^2 + n\,\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\,2x + \frac{n(n-1)(-1)^{n-3}(n-3)!\,\cancel{2}}{\cancel{2}\;x^{n-2}}$$

$$= \frac{(-1)^{n-1-2+2}(n-1)(n-2)(n-3)!}{x^{n-2}} + \frac{(-1)^{n-2-1+1}\,2n(n-2)(n-3)!}{n\,x^{n-2}}$$

$$+ \frac{(-1)^{n-3}\,n(n-1)(n-3)!}{x^{n-2}}$$

$$(-1)^{n-3}(n-3)!\,[(n-1)(n-2) + 2n(n-2) + n(n-1)]$$

$$D^n(x^2\log x) = \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}\left[n^2 - 3n + 2 - 2n^2 + 4n + n^2 - n\right]$$

$$D^n(x^2\log x) = \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}(2)$$

$$D^n(x^2\log x) = \frac{(-1)^{n-3}\,2\,(n-3)!}{x^{n-2}}$$

PAGE NO : 84 EXAMPLE : 3

If $y = \sin(m \sin^{-1}x)$, Prove that

$(1-x^2)y_2 - xy_1 + m^2 y = 0$ and

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Soln:

$y = \sin(m \sin^{-1}x)$

$\therefore \sin^{-1}y = m \sin^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = m \frac{1}{\sqrt{1-x^2}}$$

Squaring on both sides, we've

$$\frac{1}{1-y^2}\left(\frac{dy}{dx}\right)^2 = m^2 \cdot \frac{1}{1-x^2}$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$$

Differentiating the above eqn w.r.to x, we get

$$(1-x^2)\cancel{2}\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -\cancel{2}m^2y\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -m^2y$$

$$(1-x^2)y_2 - 2xy_1 + m^2y = 0 \quad \rightarrow ①$$

Taking then n^{th} derivative of each term by Leibnitz's Theorem,

We've,

$$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + {}^nc_1\frac{d^{n-1}u}{dx^{n-1}}\cdot\frac{dv}{dx} + \dots + {}^nc_r\frac{d^{n-r}u}{dx^{n-r}}\cdot\frac{d^rv}{dx^r} + \dots$$

$$ {}^nc_1\frac{du}{dx}\cdot\frac{d^{n-1}v}{dx^{n-1}} + u\cdot\frac{d^nv}{dx^n}$$

① becomes,

$$y_{n+2}(1-x^2) + {}^nc_1\,y_{n+1}(-2x) + {}^nc_2\,y_n(-2)$$

$$= y_{n+1}x + {}^nc_1\,y_n - m^2y_n$$

$$y_{n+2}(1-x^2) - 2nx\,y_{n+1} - n(n-1)y_n = x y_{n+1} + n y_n - m^2 y_n$$

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2-n^2)y_n = 0$$

TEXT BOOK PAGE NO: 85 EXCERCISE : 15

Q. NO. 1 (5)

Find the n^{th} differential co-efficient of $e^x \log x$

Soln:

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \dots + nc_{r-1} u_{n-r+1} v_{r-1} + nc_r u_{n-r} v_r + \dots + u v_n$$

$$D^n(e^x \log x) = D^n(e^x)\log x + nc_1 D^{n-1}(e^x) D(\log x) + nc_2 D^{n-2}(e^x) D^2(\log \dots + \dots + e^x D^n(\log x)$$

$$= e^x \log x + nc_1 e^x \left(\frac{1}{x}\right) + nc_2 e^x \left(-\frac{1}{x^2}\right) + \dots + e^x \frac{(-1)^{n-1}(n-1)\dots}{x^n}$$

$$= e^x \log x + nc_1 \frac{e^x}{x} - nc_2 \frac{e^x}{x^2} + \dots + e^x \frac{(-1)^{n-1}(n-1)!}{x^n}$$

TEXT BOOK PAGE NO: 85 EXCERCISE: 15

Q. NO: 1 (6)

Find The n^{th} differential coefficient of $x^3 \sin^3 x$

Soln:

$$D^n(uv) = u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \dots + nC_r u_{n-r} v_r + \dots + u v_n$$

$$D^n(\sin^3 x\, x^3) = D^n(\sin^3 x)\, x^3 + nC_1 D^{n-1}(\sin^3 x) D(x^3) + nC_2 D^{n-2}(\sin^3 x) D^2(x^3)$$

$$+ nC_3 D^{n-3}(\sin^3 x) D^3(x^3)$$

$$= \left[\frac{3}{4}\sin\left(\frac{n\pi}{2}+x\right) - \frac{3^n}{4}\sin\left(\frac{n\pi}{2}+3x\right)\right]x^3 + \quad \left[\text{By EXERCISE 13 } 1(i)\right]$$

$$nC_1\left[\frac{3}{4}\sin\left(\frac{\overline{n-1}\pi}{2}+x\right) - \frac{3^{n-1}}{4}\sin\left(\frac{\overline{n-1}\pi}{2}+3x\right)\right]3x^2 +$$

$$nC_2\left[\frac{3}{4}\sin\left(\frac{\overline{n-2}\pi}{2}+x\right) - \frac{3^{n-2}}{4}\sin\left(\frac{\overline{n-2}\pi}{2}+3x\right)\right]6x +$$

$$nC_3\left[\frac{3}{4}\sin\left(\frac{\overline{n-3}\pi}{2}+x\right) - \frac{3^{n-3}}{4}\sin\left(\frac{\overline{n-3}\pi}{2}+3x\right)\right]6$$

TEXT BOOK PAGE NO: 85 EXCERCISE: 15

Q. NO: 5

If $y = \sin^{-1} x$, Prove that $(1-x^2)y_2 - xy_1 = 0$ and

and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

Soln:

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{1/2}}$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$y = \sin^{-1} x$

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$y_2 = \frac{-1}{2(1-x^2)^{+3/2}} \cdot (-2x)$$

$$y_2 = \frac{-x}{(1-x^2)\sqrt{1-x^2}}$$

$\Rightarrow (1-x^2)y_2 = -xy_1 \Rightarrow \boxed{(1-x^2)y_2 + xy_1 = 0}$

$(1-x^2)y_2 - xy_1 = 0$

Taking then n^{th} derivative of each term by Leibnitz's The

we've,

$(1-x^2)y_{n+2} + nC_1 y_{n+1}(-2x) + nC_2 y_n(-2) - [x y_{n+1} + nC_1 y_n \cdot 1]$

$(1-x^2)y_{n+2} + n y_{n+1}(-2x) + \frac{(n)(n-1)}{2} y_n(-2) - x y_{n+1} - n y_n$

$(1-x^2)y_{n+2} - 2nx y_{n+1} \quad - n^2 y_n + n y_n - x y_{n+1} - n$

$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0$

Hence proved.

EXCERCISE : 15 PAGE NO : 86

Q. NO: 11

If $y = \frac{\log x}{x^2}$, Show that $x^3 \frac{d^3y}{dx^3} + 8x^2 \frac{d^2y}{dx^2} + 14x \frac{dy}{dx} + 4y = 0$

Soln:

$$y = \frac{\log x}{x^2}$$

$$y_1 = \frac{x^2 \left(\frac{1}{x}\right) - \log x \,(2x)}{(x^2)^2}$$

$$y_1 = \frac{x - 2x \log x}{x^4}$$

$$x \cdot y_1 = x - 2x \log x$$

$$x^4 y_1 = x - 2x \log x$$

$$x^4 y_2 + y_1 (4x^3) = 1 - 2\left[\log x + x \left(\frac{1}{x}\right)\right]$$

$$x^4 y_2 + 4x^3 y_1 = 1 - 2\log x - 2$$

$$x^4 y_2 + 4x^3 y_1 = -2\log x - 1$$

$$x^4 y_3 + y_2(4x^3) + 4[x^3 y_2 + y_1(3x^2)] = -\frac{2}{x}$$

$$x^4 y_3 + 4y_2 x^3 + 4y_2 x^3 + 12y_1 x^2 = -\frac{2}{x}$$

$\div x)$ $$x^3 y_3 + 8y_2 x^2 + 12y_1 x + \frac{2}{x^2} = 0$$

$$x^3 y_3 + 8y_2 x^2 + 12y_1 x + 2y_1 x - 2y_1 x + \frac{2}{x^2} = 0$$

$$x^3 y_3 + 8y_2 x^2 + 14y_1 x - 2x\left[\frac{x - 2x\log x}{x^4}\right] + \frac{2}{x^2} = 0$$

$$x^3 y_3 + 8y_2 x^2 + 14y_1 x - \frac{2x^2}{x^4}[1 - 2\log x] + \frac{2}{x^2} = 0$$

$$x^3 y_3 + 8y_2 x^2 + 14y_1 x - \frac{2}{x^2} + 4\frac{\log x}{x^2} + \frac{2}{x^2} = 0$$

$$x^3 y_3 + 8y_2 x^2 + 14y_1 x + 4y = 0$$

Hence proved.

HOMEWORK PROBLEMS

TEXT BOOK PAGE NUMBER	QUESTION NUMBER
84	EXEXRCISE:15 Q. NO:1(1),1(4),
85	EXERCISE:15 Q.NO:2,6
86	EXERCISE:15 Q.NO:8,10,
87	EXERCISE:15 Q.NO:16

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-10

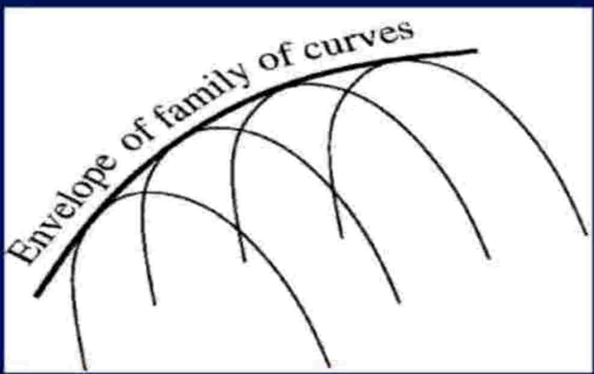
PART-4

R.RAJALAKSHMI [G.L],GACW,RAMANATHAPURAM

Description of Plane Curves

- Explicit form: $y = f(x)$
- Implicit form: $F(x, y) = 0$
- Parametric Form: $\vec{r}(t) = [x(t), y(t)]$

Families and Envelopes



Consider a family of curves. The **envelope** of the family is a curve which is tangent to every curve from the family

It is convenient to define a family of curves in implicit form

$F(x, y, a) = 0$ Where a parameterizes the family

Then the envelope is given by

$$\begin{cases} F(x, y, a) = 0 \\ \dfrac{\partial}{\partial a} F(x, y, a) = 0 \end{cases}$$

Consider the equation $x \cos\theta + y \sin\theta = a$, where a is constant. For diferent values of θ the equation represents a family of straight lines touching the circle $x^2 + y^2 = a^2$. Here θ is the parameter of the family of straight lines

$$x \cos\theta + y \sin\theta = a.$$

Similarly $y = mx + a/m$ represents a family of straight lines with the parameter m touching the parabola $y^2 = 4ax$.

Similarly $(x - a)^2 + y^2 = r^2$ where r is a constant, is a family of circles with parameter a touching the lines $y = \pm r$.

The curve E which is touched by a family of Curves C is called the *envelope* of the family of *curves* C

METHOD OF FINDING THE ENVELOPE

Let the family of curves C be $f(x,y,t) = 0$ and let us assume that a curve E, the envelope of the family exists and that its equation is $F(x,y) = 0$.

Let us also assume that for a particular value of t, say α it touches E at (ξ, η).

$\therefore f(\xi, \eta, \alpha) = 0$. (i)

and $F(\xi, \eta) = 0$ (ii)

Considering ξ, η, α independent variable and taking total differentials in (i), we have

$$\frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \eta} d\eta + \frac{\partial f}{\partial \alpha} d\alpha = 0$$

$$\text{i.e., } \frac{\partial f}{\partial \xi}\frac{d\xi}{d\alpha} + \frac{\partial f}{\partial \eta}\frac{d\eta}{\partial \alpha} + \frac{\partial f}{\partial \alpha} = 0 \qquad \text{(iii)}$$

Taking total differentials in (ii)

$$\frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta = 0$$

$$\text{i.e., } \frac{\partial F}{\partial \xi}\frac{d\xi}{d\alpha} + \frac{\partial F}{\partial \eta}\frac{d\eta}{d\alpha} = 0 \qquad \text{(iv)}$$

Since the curves $f(x,y,\alpha) = 0$ and $F(x, y) = 0$ touch one another at (ξ, η), their gradients at (ξ, η) are equal.

For $f(x, y, \alpha) = 0$, $\frac{dy}{dx}$ at $(\xi, \eta) = -\dfrac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}}$

For $F(x, y) = 0$, $\frac{dy}{dx}$ at $(\xi, \eta) = -\dfrac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}}$.

Hence $\dfrac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}} = \dfrac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}}$

Hence $$\frac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}} = \frac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}}$$

but from (iv) $$\frac{\frac{\partial F}{\partial \xi}}{\frac{\partial F}{\partial \eta}} = -\frac{\frac{\partial \eta}{\partial \alpha}}{\frac{\partial \xi}{\partial \alpha}}$$

$$\therefore \quad \frac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}} = -\frac{\frac{\partial \eta}{\partial \alpha}}{\frac{\partial \xi}{\partial \alpha}}$$

$$\therefore \quad \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial \alpha} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial \alpha} = 0 \qquad \text{(v)}$$

Comparing (iii) and (v) we get $\frac{\partial f}{\partial \alpha} = 0$ and this equation is satisfied by (ξ, η). Hence (ξ, η) satisfies both the equations $f(x,y,\alpha) = 0$ and $\frac{\partial f}{\partial \alpha} = 0$. Therefore the envelope of the family of curves $f(x,y,t) = 0$ is got by eliminating t between the equations

$$f(x,y,t) = 0 \text{ and } \frac{\partial}{\partial t} f(x,y,t) = 0.$$

§ 1.2. We sahll discuss ...

THANK YOU

when $f(x,y,t)=0$ is a Quadratic equation — Envelop is $B^2 = 4AC$

$f(x,y,t)=0$ is merely a quadratic in t, say $At^2+Bt+C=0$, where A, B, C are functions of x, y and t is the parameter, the envelope is obtained by eliminating t between the equations

$$At^2+Bt+C=0 \rightarrow (1)$$

$$2At+B=0 \rightarrow (2)$$

from (2) $t=-\frac{B}{2A}$ and substituting this value of 't' in (1)

The envelope of the eqn (1) is $\boxed{B^2 = 4AC}$

$A\left(\frac{B^2}{4A^2}\right)+B\left(-\frac{B}{2A}\right)+C=0$

$\frac{B^2-2B^2}{4A} = -C$

$-B^2 = -4AC$

$B^2 = 4AC$

EXAMPLE : 1 PAGE NO : 284

Find the envelope of the family of a straight lines $y + tx = 2at + at^3$, the parameter being 't'.

Soln:

The envelope of the family of curves $f(x, y, t) = 0$ is got by eliminating t between the equations $f(x, y, t) = 0$ and $\frac{\partial}{\partial t} f(x, y, t) = 0$

Given, $y + tx = 2at + at^3 \rightarrow (1)$

$\therefore y = -tx + 2at + at^3$

$\therefore y = t(-x + 2a + at^2) \rightarrow (2)$

Differentiate (1) partially w.r.to 't'

$x = 2a + 3at^2 \Rightarrow \boxed{t^2 = \frac{x - 2a}{3a}}$ (3)

Substituting the value of t^2 in ②, we've

② ⇒ $y = t(-x + 2a + at^2)$

$\therefore\ y = t\left(-x + 2a + a\left(\frac{x-2a}{3a}\right)\right)$

$y = \frac{t}{3}(-3x + 6a + x - 2a)$

$= \frac{t}{3}(-2x + 4a)$

$= -\frac{2t}{3}(x - 2a)$

$y = -2t\left(\frac{x-2a}{3}\right)$

$y^2 = 4t^2\left(\frac{x-2a}{3}\right)^2 = 4\cdot\left(\frac{x-2a}{3a}\right)\left(\frac{x-2a}{3}\right)^2$

$y^2 = \frac{4(x-2a)^3}{27a} \Rightarrow \underline{27ay^2 = 4(x-2a)^3}$, is the required envelope

(THIS CURVE IS CALLED A SEMI CUBICAL PARABOLA)

EXAMPLE: 2 PAGE NO: 285

Find the envelope of the family of circles $(x-a)^2+y^2=2a$, where $\underset{\smile}{a}$ is the parameter.

Soln:

To find envelope, we've to eliminate a between $f(x,y,a)=0$ and $\frac{\partial}{\partial a}f(x,y,a)=0$

Given, $(x-a)^2+y^2=2a \rightarrow ①$

Diff ① partially w.r.t 'a'

$-2(x-a)=2$

$x-a = \frac{2}{-2} = -1$

$\boxed{x-a=-1}$

$-a=-1-x$

$\boxed{a=x+1}$

Substituding $\boxed{x-a=-1}$ and $\boxed{a=x+1}$ in ① we've

$(-1)^2+y^2=2(x+1)$

$y^2+1=2x+2$

$y^2=2x+1$, is the required envelope.

Find the envelope of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{k^2-a^2} = 1$, where 'a' is the parameter.

$f(x, y, a) = 0$

$\frac{\partial f}{\partial a}(x, y, a) = 0$

Soln:

The equation of the family of curves can be written as

$$x^2(k^2-a^2) + y^2a^2 = a^2(k^2-a^2)$$

$Ax^2 + Bx + C = 0$

$$x^2k^2 - x^2a^2 + y^2a^2 - k^2a^2 + a^4 = 0$$

$a^4 - a^2(x^2-y^2+k^2) + x^2k^2 = 0$, This is the Quadratic Equation in a^2,

its eliminant is, $B^2 = 4AC$

$B = -(x^2-y^2+k^2)$, $A = 1$, $C = x^2k^2$

ie, $(x^2-y^2+k^2)^2 = 4x^2k^2$

$$x^2 - y^2 + k^2 = \pm 2xk$$

$$y^2 = x^2 \pm 2xk + k^2$$

$$y^2 = (x \pm k)^2$$

$$\pm y = x \pm k$$

$\therefore x \pm y = \pm k$, which is the required envelope.

EXAMPLE : 4 PAGE NO: 2.86

Find the envelope of the circles drawn on the radius vector of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameter.

Soln:

The coordinates of any point P on the ellipse are $(a\cos\theta, b\sin\theta)$

The equation of the circle on CP as diameter is

$$x(x - a\cos\theta) + y(y - b\sin\theta) = 0$$

ie, $x^2 + y^2 - ax\cos\theta - by\sin\theta = 0 \rightarrow (1)$

Diff (1) partially w.r.to 'θ', we've

$$ax\sin\theta - by\cos\theta = 0 \rightarrow (2)$$

Eliminate 'θ' between (1) and (2),

from (2) $\dfrac{\cos\theta}{ax} = \dfrac{\sin\theta}{by} = \dfrac{1}{\sqrt{a^2x^2 + b^2y^2}}$

$\therefore \cos\theta = \dfrac{ax}{\sqrt{a^2x^2+b^2y^2}}$ & $\sin\theta = \dfrac{by}{\sqrt{a^2x^2+b^2y^2}}$

Subs, $\sin\theta$ & $\cos\theta$ in (1), we've

$\dfrac{ax}{\cos\theta} = \dfrac{by}{\sin\theta}$

$\dfrac{a^2x^2}{\cos^2\theta} = \dfrac{b^2y^2}{\sin^2\theta}$

$a^2x^2 + b^2y^2$

$= \dfrac{b^2y^2}{\sin^2\theta}\cos^2\theta + b^2y^2$

$= \dfrac{b^2y^2}{\sin^2\theta}(\cos^2\theta + \sin^2\theta)$

$a^2x^2 + b^2y^2 = \dfrac{b^2y^2}{\sin^2\theta}$

$\dfrac{1}{a^2x^2+b^2y^2} = \left(\dfrac{\sin\theta}{by}\right)^2$

$\dfrac{\sin\theta}{by} = \pm\dfrac{1}{\sqrt{a^2x^2+b^2y^2}}$

$$x^2+y^2-\frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2}}-\frac{b^2y^2}{\sqrt{a^2x^2+b^2y^2}}=0$$

$$x^2+y^2=\frac{a^2x^2+b^2y^2}{\sqrt{a^2x^2+b^2y^2}}$$

$$x^2+y^2=\sqrt{a^2x^2+b^2y^2}$$

$\therefore (x^2+y^2)^2=a^2x^2+b^2y^2$ which is the eqn of the required envelope

EXAMPLE: 5 PAGE NO: 287 (

[Equations having Two parameter]

Find the envelope of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters are related by the equation $a^2+b^2=c^2$ where 'c' is a constant.

Given $a^2+b^2=c^2 \rightarrow ①$ Let a & b as fns of 't'

$\frac{x}{a}+\frac{y}{b}=1 \rightarrow ②$

Diff ② w.r.to 't'

$-\frac{x}{a^2}\cdot\frac{da}{dt}-\frac{y}{b^2}\frac{db}{dt}=0 \rightarrow ③$

Diff ① w.r.to 't'

$2a.\frac{da}{dt}+2b\frac{db}{dt}=0 \rightarrow ④$

comparing ③ & ④

$-\frac{x}{a^2}=2a \Rightarrow \frac{-x}{a^3}=2 \Rightarrow \frac{x}{a^3}=-2$

and $-\frac{y}{b^2}=2b \Rightarrow -\frac{y}{b^3}=2 \Rightarrow \frac{y}{b^3}=-2$

$\frac{x}{a^3}=\frac{y}{b^3} \rightarrow ⑤$

from ⑤

$$\frac{\left(\frac{x}{a}\right)}{a^2} = \frac{\left(\frac{y}{b}\right)}{b^2} = \frac{\left(\frac{x}{a} + \frac{y}{b}\right)}{(a^2+b^2)} = \frac{1}{c^2}$$

$$\frac{x}{a^3} = \frac{1}{c^2} \Rightarrow a^3 = c^2 x \Rightarrow a = (c^2 x)^{1/3}$$

$$\frac{y}{b^3} = \frac{1}{c^2} \Rightarrow b^3 = c^2 y \Rightarrow b = (c^2 x)^{1/3}$$

Substituting the above in ①

$$① \Rightarrow a^2 + b^2 = c^2$$

$$\Rightarrow (c^2 x)^{2/3} + (c^2 y)^{2/3} = c^2$$

$$\Rightarrow c^{4/3}\left[x^{2/3} + y^{2/3}\right] = c^2 \Rightarrow x^{2/3} + y^{2/3} = c^{2(1/3)}$$

$x^{2/3} + y^{2/3} = c^{2/3}$ is the required envelope.

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-10[2.1-2.3]

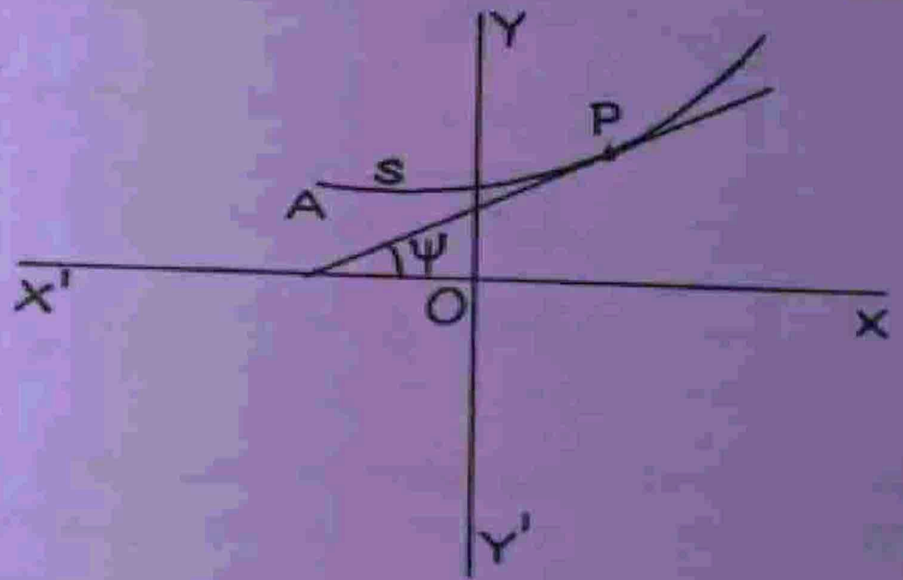
PART-6

R.RAJALAKSHMI [G.L],GACW,RAMANATHAPURAM

; (*ii*) a + b = c

Curvature.

§ 2.1. A curve has a definite direction at every point on it. At any particular point, the direction of the curve is the same as that of the tangent to the curve at that point. The direction usually



changes from point to point and the tangent line rotates as the point moves along the curve.

Let s denote the length of arc AP measured from some fixed point A on the curve and ψ the angle which the tangent makes with the x - axis. As P moves along the curve s and ψ vary and the rate at which ψ increases relative to s i.e., $\frac{d\psi}{ds}$ is called the curvature of the curve at the point P. Its value does not depend on the position of the point A or of the line Ox from which s and ψ are measured but its sign depends on the sense in which s is measured. From the definition of curvature at P, it is easuly seen that the curvature is the rate of change of the direction of the tangent at P. Roughly, we can say that the curvature is the rate at which the curve 'curves' and its sign indicates the direction in which the tangent is turning as s increases.

... radius r. Draw AO the tangent

as s increases

Let ABC be any given circle of radius r. Draw AQ the tangent at A. O be the centre of the circle. Join OA. Select any point P on the circle. Draw PM the tangent at P cutting. AQ at an angle ψ. Measure the length of the arc of the circle from A, so that the arc AP is s.

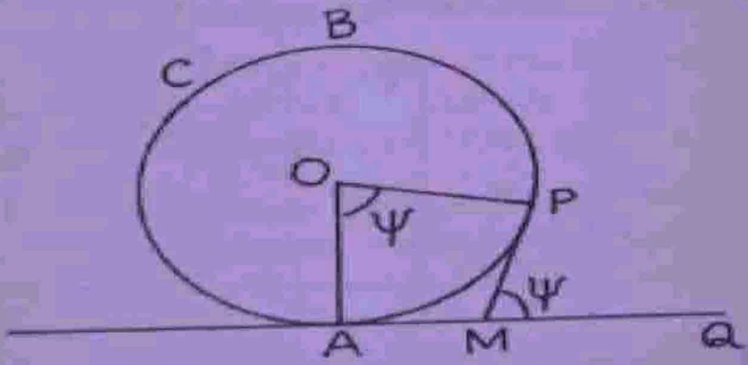


Fig. 29

As ∠ AOP $= \psi$,

$s = r\psi$ (r is constant, being the radius of the circle)

$\frac{ds}{d\psi} = r \quad \therefore \quad \frac{d\psi}{ds} = \frac{1}{r}$

Curvature of the circle is reciprocal of its radius

... circle is the reciprocal of its radius.

§ 2.2. Circle, radius and centre of curvature.

Let P and Q be two points on a plane curve, ψ and $\psi + \Delta\psi$ the angles which the tangents at P and Q make with the x-axis ; s the arc measured from some fixed point A on the curve up to P and Δs the arc PQ. Let the normals at P,Q intersect at C.

From the figure, it easily follows that $\angle PC'Q = \Delta\psi$.

$$\frac{PC'}{\sin PQC'} = \frac{\text{chord } PQ}{\sin PC'Q}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\sin PC'Q}$$

$$= \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right) \cdot \frac{\Delta s}{\sin \Delta\psi}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\Delta s}{\Delta\psi} \cdot \frac{\Delta\psi}{\sin \Delta\psi}$$

Now the limit of $\angle PQC'$ as Q tends to P is 90° and also

$$\underset{Q \to P}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1, \quad \underset{\Delta\psi \to 0}{Lt} \frac{\Delta s}{\Delta \psi} = \frac{ds}{d\psi} \text{ and } \underset{\Delta\psi \to 0}{Lt} \frac{\sin \Delta\psi}{\Delta \psi} = 1.$$

$\therefore$ As Q tends to P, limit of PC' is $\frac{ds}{d\psi}$.

Let the limiting position of C' be C.

Then $PC = \frac{ds}{d\psi}$, *i.e.*, $\frac{1}{PC} = \frac{d\psi}{ds}$.

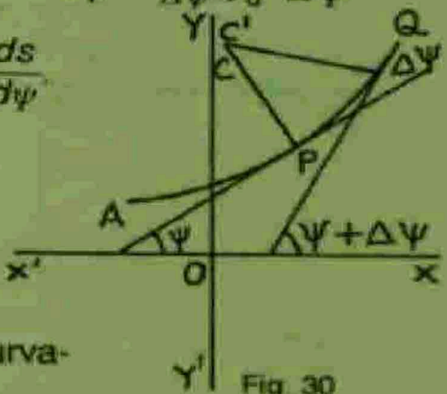


Fig. 30

$PC = \frac{ds}{d\psi}$ is called the radius of curvature at P.

The circle whose centre is C and radius PC has therefore the same tangent and the same curvature as the curve has at P.

This circle is called the *circle of curvature* at P. So it can be defined as that circle which touches the given curve at the point, has a radius equal to the radius of curvature at the point and lies on the same side of the tangent as the curve. Its radius is PC, the radius of curvature and its centre is C, the centre of curvature at the point P. The radius of curvature is often denoted by ρ and so the curvature is $\frac{1}{\rho}$

§ 2.3. **Cartesian formula for the radius of curvature.**

We know that $\frac{dy}{dx} = \tan\psi$.

$$\therefore \frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx} = \sec^2\psi \frac{d\psi}{ds}\frac{ds}{dx}$$

$$\therefore \frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} \text{ as } \frac{dx}{ds} = \cos\psi \text{ by §3 in Chapter IX}$$

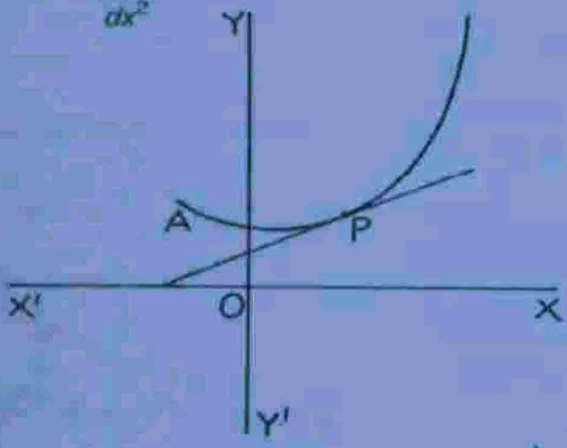


Fig. 31

$$= \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Examples.

Ex. 1. What is the radius of curvature of the curve

$x^4 + y^4 = 2$ at the point (1,1) ? (B.Sc. 1990)

Differentiating the above equation, we get

$$4x^3 + 4y^3 \frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{x^3}{y^3}$$

Differentiating this once again, we get

$$\frac{d^2y}{dx^2} = \frac{3\left(x^3 \frac{dy}{dx} - x^2 y\right)}{y^4}.$$

At the point (1,1), $\frac{dy}{dx} = -1$, and $\frac{d^2y}{dx^2} = -6$

$$\therefore \rho = \frac{(1+1)^{3/2}}{-6} = -\frac{\sqrt{2}}{3}.$$

Ex. 2. Show that the radius of curvature at any point of the catenary $y = c\cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

(B.Sc. 1990)

Here $\frac{dy}{dx} = \sinh\frac{x}{c}$ and hence

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \left(1 + \sinh^2\frac{x}{c}\right)^{3/2} = \cosh^3\frac{x}{c}.$$

Also $\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}.$

Here $\rho = \dfrac{\cosh^3\frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}} = c\cosh^2\frac{x}{c} = \frac{y^2}{c}.$

Again at any point (x,y)

the normal $= y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = y\cosh\frac{x}{c} = \frac{y^2}{c}.$

$\therefore$ Radius of curvature = length of the normal.

∴ Radius of curvature = length of the normal.

Ex. 3. If a curve is defined by the parametric equation $x = f(\theta)$ and $y = \phi(\theta)$, prove that the curvature is

$$\frac{1}{\rho} = \frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}},$$

where dashes denote diffrentiation with respect to θ.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{y'}{x'}.$$

$$\frac{d^2 y}{d x^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{d\theta}\left(\frac{y'}{x'}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{y'' x' - y' x''}{x'^2} \frac{1}{x'}$$

$$= \frac{y'' x' - y' x''}{x'^3}.$$

$$\therefore \frac{1}{\rho} = \frac{\frac{d^2 y}{d x^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{y'' x' - y' x''}{x'^3 \left[1 + \frac{y'^2}{x'^2}\right]^{3/2}}$$

$$= \frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}}$$

Ex. 4 Prove ...

$(x'^2 + y'^2)^{3/2}$

Ex. 4. Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ is $4a \cos\dfrac{\theta}{2}$.

(B.Sc. 1988)

From the given equations,

$$\frac{dx}{d\theta} = a(1 + \cos\theta);\ \frac{dy}{d\theta} = a\sin\theta.$$

$$\frac{d^2x}{d\theta^2} = -a\sin\theta;\ \frac{d^2y}{d\theta^2} = a\cos\theta.$$

Substituting the values in the formula obtained in the previous example, we get

$$\frac{1}{\rho} = \frac{a(1 + \cos\theta)\, a\cos\theta - a\sin\theta(-a\sin\theta)}{\left[a^2(1 + \cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}$$

$$= \frac{a^2(1 + \cos\theta)}{a^3[2(1 + \cos\theta)]^{3/2}}$$

$$= \frac{2\cos^2\theta/2}{a[4\cos^2\theta/2]^{3/2}} = \frac{1}{4a\cos\theta/2}$$

$$\therefore\ \rho = 4a\cos\frac{\theta}{2}.$$

Ex. 5. Find ρ at the point 't' of the curve

$$x = a(\cos t + t \sin t)\ ;\ y = a(\sin t - t\cos t).$$

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t.$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t.$$

$$\therefore \frac{dy}{dx} = \tan t.$$

Differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t)\frac{dt}{dx} = \sec^2 t\,\frac{1}{at\cos t} = \frac{1}{at\cos^3 t}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1 + \tan^2 t)^{3/2}}{\frac{1}{at\cos^3 t}} = at.$$

HOMEWORK PROBLEMS

TEXT BOOK PAGE NUMBER	QUESTION NUMBER
298	EXEXRCISE43 Q.NO:1
299	EXERCISE:43 Q.NO:2(ii),3(i),(ii),(iii),4
300	EXERCISE:43 Q.NO:5,8,9
301	EXERCISE:43 Q.NO:10,16,17,18,25

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-10[2.4]

PART-7

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§ 2.4. The coordinates of the centre of curvature.

Let the centre of curvature of the curve $y = f(x)$ corresponding to the point P (x,y) be X and Y.

$$X = ON$$
$$= OQ - NQ = OQ - MP$$
$$= x - PC \sin\psi = x - \rho \sin\psi.$$
$$Y = NC = NM + MC$$
$$= QP + PC \cos\psi = y + \rho \cos\psi.$$

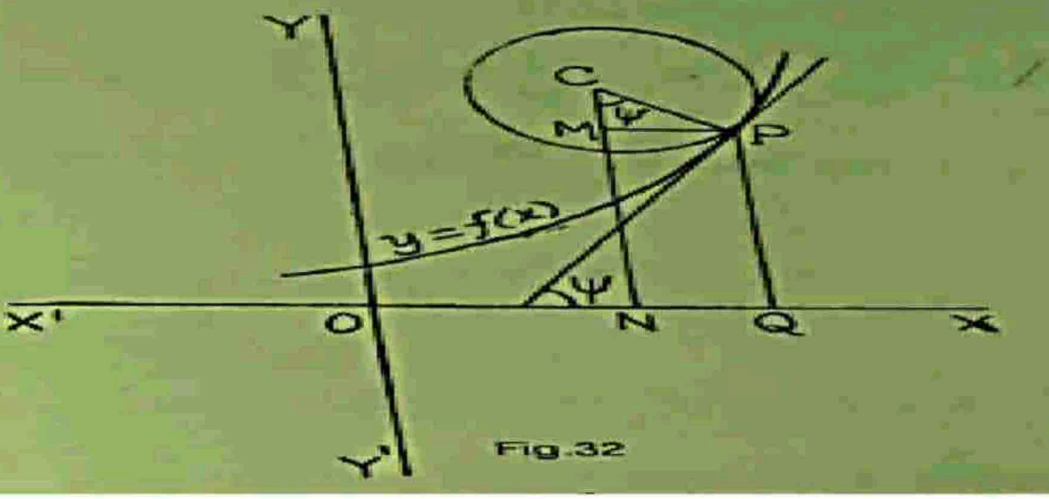

Fig.32

If y_1 and y_2 denote $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ we know that

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \text{ and } \tan\psi = y_1$$

$$\therefore \cos\psi = \frac{1}{\sqrt{1 + y_1^2}} \text{ and } \sin\psi = \frac{y_1}{\sqrt{1 + y_1^2}}$$

$$\therefore X = x - \frac{(1 + y_1^2)^{3/2}}{y_2} \frac{y_1}{(1 + y_1^2)^{1/2}} = x - \frac{y_1(1 + y_1^2)}{y_2}$$

$$Y = y + \frac{(1 + y_1^2)^{3/2}}{y_2} \frac{1}{(1 + y_1^2)^{1/2}} = y + \frac{(1 + y_1^2)}{y_2}.$$

The locus of the centre of curvature for a curve is called the evolute of the curve.

Examples.

✓ **Ex. 1.** Find the co-ordinates of the centre of curvature of the curve $xy = 2$ at the point (2, 1).

Here $y = \frac{2}{x}$, $\frac{dy}{dx} = -\frac{2}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{4}{x^3}$.

$\therefore$ At (2, 1) the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are respectively $-1/2$ and $1/2$.

$$\therefore X = 2 + \frac{(1 + \frac{1}{4}) \times \frac{1}{2}}{\frac{1}{2}} = 3\frac{1}{4}.$$

$$Y = 1 + \frac{1 + 1/4}{1/2} = 3\frac{1}{2}.$$

$\therefore$ The centre of curvature is $(3\frac{1}{4}, 3\frac{1}{2})$.

Ex. 2. Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a(1+t^2)^{3/2}$, $X = 2a + 3at^2$, $Y = -2at^3$. Deduce the equation of the evolute. (B.Sc. 1986)

$$x = at^2,\ y = 2at.$$

$$\therefore \frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a;\ \frac{dy}{dx} = \frac{1}{t}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \div \frac{dx}{dt}$$

$$= -\frac{1}{t^2} \div 2at = -\frac{1}{2at^3}.$$

$$\therefore \rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \div \frac{d^2y}{dx^2} = -2a(1+t^2)^{3/2}.$$

$$X = x - \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}} = at^2 - \frac{\left(1 + \frac{1}{t^2}\right)\frac{1}{t}}{-\frac{1}{2at^3}}$$

$$= 2a + 3at^2.$$

$$Y = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = 2at + \frac{1 + \frac{1}{t^2}}{-\frac{1}{2at^2}} = -2at^3.$$

Eliminating t from X and Y,

$$Y = -2a\left(\frac{X - 2a}{3a}\right)^{3/2}$$

Squaring both sides and simplifying, we get

$$27aY^2 = 4(X - 2a)^3.$$

The locus of (X, Y) is $27ay^2 = 4(x - 2a)^3$.

The curve is called a semi-cubical parabola.

Ex. 3. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(B.Sc. 1993) (B.Sc. 1990)

Any point on the ellipse is $(a \cos\theta, b \sin\theta)$.

$$x = a\cos\theta;\ \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta;\ \frac{dy}{d\theta} = b\cos\theta.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right) = \frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\frac{b}{a^2}\ cosec^3\theta$$

$$X = x - \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

$$= a\cos\theta - \frac{\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)\left(\frac{b}{a}\cot\theta\right)}{\frac{b}{a^2}\ cosec^3\theta}$$

$$= \frac{(a^2 - b^2)\cos^3\theta}{a}$$

$$Y = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2 y}{d x^2}}$$

$$= b \sin\theta - \frac{1 + \frac{b^2}{a^2}\cot^2\theta}{\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= -\frac{a^2 - b^2}{b}\sin^3\theta.$$

$$\cos\theta = \left(\frac{aX}{a^2 - b^2}\right)^{1/3} ; \sin\theta = \left(\frac{-bY}{a^2 - b^2}\right)^{1/3}$$

To eliminate θ, squaring and adding, we get

$$\left(\frac{aX}{a^2 - b^2}\right)^{2/3} + \left(\frac{-bY}{a^2 - b^2}\right)^{2/3} = 1.$$

$$\text{i.e., } \left(\frac{aX}{a^2 - b^2}\right)^{2/3} + \left(\frac{bY}{a^2 - b^2}\right)^{2/3} = 1$$

$\therefore$ The locus of (X, Y) is the four cusped hypocycloid.

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

Ex. 4. Show that the evolute of the cycloid

$x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$ is another cycloid.

(B.Sc. 1994) (B.Sc. 1990)

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \frac{dy}{dx} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \cot\frac{\theta}{2}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right) = -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \cdot \frac{d\theta}{dx}.$$

$$= -\frac{1}{4a\sin^4\frac{\theta}{2}}.$$

$$X = x + \frac{\left(1 + \cot^2\frac{\theta}{2}\right)\cot\frac{\theta}{2}}{\dfrac{1}{4a\sin^4\frac{\theta}{2}}}.$$

$$= a(\theta - \sin\theta) + 2a\sin\theta.$$

$$= a(\theta + \sin\theta)$$

$$Y = y + \frac{1 + \cot^2\frac{\theta}{2}}{-\frac{1}{4a\sin^4\frac{\theta}{2}}}$$

$$= a(1 - \cos\theta) - 2a(1 - \cos\theta)$$

$$= -a(1 - \cos\theta)$$

$\therefore$ The locus of (X,Y) is

$$x = a(\theta + \sin\theta)\ ;\ y = -a(1 - \cos\theta)$$

This is also a cycloid.

HOMEWORK PROBLEMS

TEXT BOOK PAGE NUMBER	QUESTION NUMBER
308	EXEXRCISE44 Q. NO:1,3,4,5,8
309	EXERCISE:434Q.NO:10

THANK YOU

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-1

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-10[2.6-2.7]

PART-8

R.RAJALAKSHMI [G.L],GACW,RAMANATHAPURAM

§ 2.6. Radius of curvature when the curve is given in polar co-ordinates.

Let us assume that the equation of the curve in polar coordinates be $r = f(\theta)$.

In the figure .

$$\psi = \theta + \phi.$$

$$\therefore \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}.$$

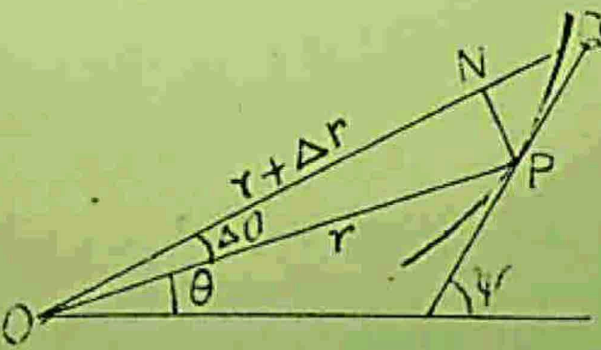

Fig.33

We have proved that

$$\tan\varphi = r\frac{d\theta}{dt} = \frac{r}{\left(\frac{dr}{d\theta}\right)}.$$

Differentiating w.r.t θ, we get

$$\sec^2\varphi \frac{d\varphi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\therefore \frac{d\varphi}{d\theta} = \frac{1}{\sec^2\varphi} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1 + \frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}.$$

$$= \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\psi}{d\theta} = 1 + \frac{d\varphi}{d\theta}$$

$$= 1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$= \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

We have proved in the previous chapter that

$$\frac{ds}{d\theta} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}$$

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\psi}$$

$$= \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2} \frac{\left(\frac{dr}{d\theta}\right)^2 + r^2}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

Examples

Examples.

Ex. 1. Find the radius of curvature of the cardioid

$$r = a(1 - \cos\theta).$$ (B.Sc. 1985)

Here $\frac{dr}{d\theta} = a\sin\theta$, $\frac{d^2r}{d\theta^2} = a\cos\theta$.

$$\therefore \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = \left[a^2(1-\cos\theta)^2 + (a^2\sin^2\theta)\right]^{3/2}$$

$$= 8a^3\sin^3\frac{\theta}{2}.$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)$$

$$= 6a^2\sin^2\frac{\theta}{2}.$$

$$\therefore \rho = \frac{8a^3\sin^3\frac{\theta}{2}}{6a^2\sin^2\frac{\theta}{2}} = \frac{4}{3}a\sin\frac{\theta}{2}$$

$$= \frac{2}{3}\sqrt{2ar}$$

Ex. 2. Show that the radius of curvature of the curve

$$r^n = a^n \cos n\theta \quad \text{is} \quad \frac{a^n r^{-n+1}}{n+1}.$$

Taking logatithms on both sides and differentiating, we get

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}$$

$$\therefore \frac{dr}{d\theta} = -r \tan n\theta.$$

Differentiating once again w.r.t. θ, we get

$$\frac{d^2 r}{d\theta^2} = -\frac{dr}{d\theta}\tan n\theta - nr \sec^2 n\theta$$

$$= r\tan^2 n\theta - nr \sec^2 n\theta.$$

$$\rho = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{(n+1) r^2 \sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta}$$

$$= \frac{r \cdot a^n}{(n+1) r^n} = \frac{a^n r^{-n+1}}{n+1}.$$

Particular cases.

(i) Putting $n = 2$, we get Bernouilli's lemniscate;

$$\rho = \frac{a^2}{3r}.$$

(ii) When $n = -2$, we have a recangular hyperbola ;

$$\rho = \frac{r^3}{a^2}.$$

(iii) When n = 1/2, we get cardioid ; $\rho = \frac{2}{3}\sqrt{ar}$.

(iv) When $n = -1/2$, we get a parabola ; $\rho = \frac{2r^{3/2}}{\sqrt{a}}$.

(v) When $n = 1$. we get a circles ; $\rho = \frac{a}{2}$.

§ 2.7. p – r equation ; pedal equation of a curve.

Let OA be the initial line, O the pole and P any point on the curve. The length of the perpendicular drawn from the pole on the tangent at the point P is usually denoted by p.

$$OY = p = r \sin O\hat{P}Y = r \sin (180 - \varphi) = r \sin \varphi$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \varphi}$$

$$= \frac{1}{r^2}(1 + \cot^2 \varphi)$$

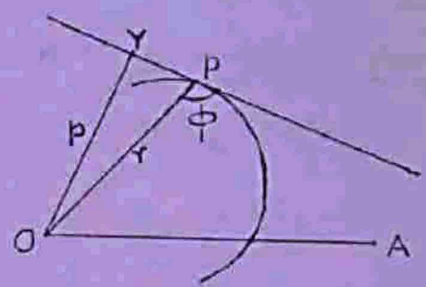

Fig. 34

$$= \frac{1}{r^2}\left\{1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right\}.$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

If $r = \frac{1}{u}$, $\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

$$\therefore \frac{1}{p^2} = u^2 + u^4\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2$$

$$= u^2 + \left(\frac{du}{d\theta}\right)^2$$

Cor. For any curve $\frac{ds}{d\theta} = \frac{r^2}{p}$.

Examples.

Ex. 1. Prove that the $(p - r)$ equation of the cardioid

$r = a(1 - \cos\theta)$ is $p^2 = \frac{r^3}{2a}$. (B.Sc. 1994) (B.Sc. 1990)

$r = a(1 - \cos\theta)$. $\therefore \frac{dr}{d\theta} = a \sin\theta$.

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4} a^2 \sin^2\theta$$

$$= \frac{r^2 + a^2\sin^2\theta}{r^4} = \frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{r^4}$$

$$= \frac{2a^2 - 2a^2\cos\theta}{r^4} = \frac{2a^2(1-\cos\theta)}{r^4}$$

$$= \frac{2ar}{r^4} = \frac{2a}{r^3}.$$

$$p^2 = \frac{r^3}{2a}.$$

Ex. 2. From the polar equation of the parabola, show that

$$p^2 = ar. \qquad \text{(B.Sc. 1989)}$$

Polar equation of the parabola is $\frac{2a}{r} = 1 - \cos\theta$ (with respect to the focus as pole).

Differentiating both sides with respect to θ, we get

$$-\frac{2a}{r^2}\frac{dr}{d\theta} = \sin\theta.$$

$$\therefore \frac{1}{r^2}\frac{dr}{d\theta} = -\frac{\sin\theta}{2a}.$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{\sin^2\theta}{4a^2}$$

$$= \frac{(1-\cos\theta)^2}{4a^2} + \frac{\sin^2\theta}{4a^2} = \frac{2-2\cos\theta}{4a^2}$$

$$= \frac{2a}{r}\cdot\frac{2}{4a^2} = \frac{1}{ar}.$$

$$\therefore p^2 = ar.$$

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-2

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-11[1-7]

PART-9

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CHAPTER XI

LINEAR ASYMPTOTES

§ 1. **Definition.** If a straight line cuts a curve in two points at an infinite distance from the origin, itself not lying wholly at infinity, it is called an asymptote to the curve.

§ 2. To find the equtions of the asymptotes of a plane algebraic curve.

Let the equation of any curve of the n^{th} degree be arranged in homgeneous sets of terms. Then it can be written as

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \quad = 0, \qquad (1)$$

where $\varphi_n\left(\frac{y}{x}\right)$ is an expression of the n^{th} degree in $\left(\frac{y}{x}\right)$, etc.

Let us find where the straight line $y = mx + c$ cuts the curve.

Putting $\frac{y}{x} = m + \frac{c}{x}$ in (1), we have

$$x^n \phi_n\left(m + \frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m + \frac{c}{x}\right) + \ldots = 0,$$

giving the abscissae of the points of intersection.

Expanding each of the terms by Taylor's theorem, we have

$$x^n \phi_n(m) + x^{n-1}\left[c\,\phi_n'(m) + \phi_{n-1}(m)\right]$$
$$+ x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m)\right] \ldots = 0 \quad (2)$$

This is an equation of the nth degree in x ; hence the straight line $y = mx + c$ will cut the curve (1) in n points real or imaginary. If $\phi_n(m)$, the coefficient of the highest power of x be zero, then one root of (2) is infinite : if further we equate coefficient of x^{n-1} in (2) to zero, viz.,

$c\,\phi_n'(m) + \phi_{n-1}(m) = 0$, then a second root of (2) is infinite.

In other words, $y = mx + c$ will be an asymptote if

$$\phi_n(m) = 0 \qquad \text{(i)}$$

and $$c\,\phi_n'(m) + \phi_{n-1}(m) = 0 \qquad \text{(ii)}$$

Equation (i), being of the nth degree, gives n values for m say $m_1, m_2, \ldots, m_n$.

The corresponding values of c are got from (ii) as

$$c_1 = -\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)},\; c_2 = -\frac{\phi_{n-1}(m_2)}{\phi_n'(m_2)}, \ldots \text{ etc.}$$

The n asympototes of the curve (1) are therefore

$$y = m_1 x + c_1$$

$$y = m_2 x + c_2$$

....

$$y = m_n x + c_n$$

$$c_1 = -\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)},\ c_2 = -\frac{\phi_{n-1}(m_2)}{\phi_n'(m_2)}, \ldots \text{ etc.}$$

The n asympototes of the curve (1) are therefore

$y = m_1 x + c_1$

$y = m_2 x + c_2$

....

$y = m_n x + c_n$

Rule. In the highest degree terms, put $x = 1$ and $y = m$; this gives $\phi_n(m) = 0$; hence m is found. Form $\phi_{n-1}(m)$, in a similar way from terms of degree $n - 1$ and differentiate $\phi_n(m)$, then the values of c are got from the formula $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$, by putting

$m = m_1 \ldots\ldots m_n$.

Cor. 1. As $\phi_n(m) = 0$ is of the nth degree, there are n values of m; hence there exist n asymptotes, real or imaginary for a curve of the n^{th} degree.

Cor. 2. If the degree of an equation be odd, there exists at least one real root as imaginary roots occur in pairs only. Hence no curve of an odd degree can be closed. A curve of odd degree cannot have an even number of real asymptotes.

Cor. 2. If the degree of an equation be odd, there exists at least one real root as imaginary roots occur in pairs only. Hence no curve of an odd degree can be closed. A curve of odd degree cannot have an even number of real asymptotes.

Cor. 3. If the term y^n be absent in the equation of the curve, then the term m^n will be missing in $\phi_n(m) = 0$. Hence the degree of this equation is apparently reduced by one. But we know that one value of m is ∞. Therefore, the corresponding asymptote is perpendicular to the axis of x, i.e., parallel to the y-axis. This naturally leads up to the consideration of asymptotes parallel to either axis.

either axis.

§ 3. Asymptotes parallel to the axis.

If the equation of the curve be arranged in the form

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$$
$$+ b_1 x^{n-1} + b_2 x^{n-2} y + b_n y^{n-1}$$
$$+ c_2 x^{n-2} + \dots = 0,$$

rearranging in decending powers of x, we have

$$a_0 x^n + (a_1 y + b_1) x^{n-1} + \dots = 0 \qquad (1)$$

If $a_0 = 0$ and y be chosen to satisfy $a_1 y + b = 0$, the two highest powers of x in equation (1) vanish and therefore two of its roots are infinite. Hence $a_1 y + b = 0$ is an asymptote.

Similarly if $a_n = 0$, $a_{n-1} x + b = 0$ is an asymptote.

Hence the rule to find asymptotes parallel to the axes is :

Equate to zero the coefficients of the highest powers of x and y.

Example.

The asymptotes of $x^2y^2 = c^2(x^2 + y^2)$ are the sides of a square.

(B.Sc. 1994) (B.Sc. 1980)

Equating the coefficient of the highest power of x, i.e., that of x^2, viz., $y^2 - c^2 = 0$, we get $y = \pm c$ as two asymptotes parallel to the x-axis. Similarly the coefficient of y^2, i.e., $x^2 - c^2 = 0$ gives $x = \pm c$, two asymptotes parallel to the y-axis. Obviously the four asymptotes $x = \pm c$, $y = \pm c$ form the sides of the square.

Ex. 1. Find the asymptotes of the cubic

$$y^3 - 6xy^2 + 11x^2 y - 6x^3 + x + y = 0.$$

The highest degree terms, i.e., the third degree terms are

$$y^3 - 6xy^2 + 11x^2 y - 6x^3$$

By the rule, put $x = 1$ and $y = m$; we get

$$\phi_3 (m) = m^3 - 6m^2 + 11m - 6 = 0.$$

Solving $m = 1, m = 2, m = 3$. $\phi_{n-1} (m) = 0$ as there are no second degree terms.

Hence $c = -\dfrac{\phi_{n-1} (m)}{\phi_n' (m)} = 0.$

$\therefore$ The three asymptotes are $y = x$, $y = 2x$ and $y = 3x$.

Ex. 2. Find the asymptotes of

$$x^3 + 2x^2 y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0.$$

Here the highest degree terms are $x^3 + 2x^2 y - xy^2 - 2y^3$.

Putting $x = 1$ and $y = m$,

$$\phi_3 (m) = 1 + 2m - m^2 - 2m^3 = 0.$$

$\therefore$ $2m^3 - 2m + m^2 - 1 = 0$ or $(m^2 - 1)(2m + 1) = 0.$

$$m = \pm 1, -\frac{1}{2}.$$

$$\phi_2 (m) = 4m^2 + 2m.$$

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m(2m+1)}{2(1-m-3m^2)} = +\frac{m(2m+1)}{3m^2+m-1}.$$

For $m = 1, c = 1$: the corresponding asymptote is $y = x + 1$.

For $m = -1, c = 1$: " $y = -x + 1$.

For $m = -\frac{1}{2}, c = 0$: " $y = -\frac{x}{2}$.

Ex. 3. Find the rectilinear asymptotes of the curve

$$y^2(x^2 - y^2) - 2ay^3 + 2a^3x = 0.$$

Following the rule of § 2.

$$\varphi_4(m) = m^2(1 - m^2) = 0$$

$$\therefore m = 0, 1, -1.$$

$$c = -\frac{\varphi_3(m)}{\varphi_4'(m)} = +\frac{2am^3}{2m(1 - 2m^2)} = \frac{am^2}{1 - 2m^2}.$$

For $m = 0$, $c = 0$; the corresponding asymptote is $y = 0$.

For $m = 1$, $c = -a$; " $y = x - a$.

For $m = -1$, $c = -a$; " $y = -x - a$.

[**Note** : Equating the coefficient of x^2, i.e., $y^2 = 0$, the asymptote $y = 0$ could have been deduced by rule of 3.]

§ 4. Special cases.

Let us consider the equations $\phi_n(m) = 0$ ....(1) and

$c\phi_n'(m) + \phi_{n-1}(m) = 0$.....(2) of § 2.

Case 1. Suppose the roots $m_1, m_2, \ldots, m_n$ of $\phi_n(m) = 0$ are all different so that $\phi_n'(m) \neq 0$ for any of these roots.

Let $\phi_n(m) = 0$ and $\phi_{n-1}(m) = 0$ have a common factor say $m - m_1$; then $c_1 = 0$. The corresponding asymptote is $y = m_1 x$ passing through the origin.

Case 2. Suppose two roots m_1 and m_2 of $\phi_n(m) = 0$ are equal ; then $\phi_n'(m_1) = 0$. If $\phi_{n-1}(m_1) \neq 0$, c as determind from (2) is infinite. The line $y = m_1 x + c_1$ meets the curve in two points at infinity and makes an infinite intercept along the y-axis. Hence it lies wholly at infinity. Strictly speaking it cannot be considered to be an asymptote but is counted as one of the n theoretical asymptotes of the given curve.

Case 3. $\phi_n(m) = 0$ has two equal roots, each to m_1 say ; then $\phi_n'(m_1) = 0$. If m_1 also satisfies $\phi_{n-1}(m_1) = 0$, c cannot be determined from (2). So we resort to the coefficient of x^{n-2} in (2) of § 2 and make it satisfy.

$$\frac{c^2}{2}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0 \qquad \text{(I)}$$

Let the two roots of this equation be c_1, c_2, real or imaginary.

Then the two corresponding asymptotes $y = m_1 x + c_1$ and $y = m_1 x + c_2$ are parallel so that each cuts the curve in three points of infinity.

Any straight line parallel to $y = m_1 x$ will meet the curve in two points at infinity. Of this family of parallel lines, we pick those cutting the curve in three points at infinity. They only are termed the asymptotes of the curve.

The combined equation of these two asymptotes is got from (i) by putting $y - m_1 x$ for c.

$$\text{i.e., } (y - m_1 x)^2 \phi_n''(m_1) + 2(y - m_1 x)\phi'_{n-1}(m_1) + 2\phi_{n-2}(m_1) = 0.$$

Example.

Find the asymptotes of

$$x^3 + 2x^2 y - 4xy^2 - 8y^3 - 4x + 8y = 1. \quad \text{(B.Sc. 1986)}$$

Proceeding as per rule of § 2.

$$\phi_n(m) = 1 + 2m - 4m^2 - 8m^3 = 0$$

$$\text{i.e., } (1 - 2m)(1 + 2m)^2 = 0.$$

$$\therefore \; m_1 = -\frac{1}{2} = m_2 \text{ and } m_3 = \frac{1}{2}.$$

c is determined by $c\,\phi_n'(m) + \phi_{n-1}(m) = 0$

i.e., $c[2 - 8m - 24m^2] = 0$ as $\varphi_{n-1}(m) = 0$ here (1)

For $m_3 = \frac{1}{2}$, $c\,[2 - 4 - 6] = -8c = 0$, $c = 0$.

Corresponding asymptote is $y = \frac{1}{2}x$, i.e., $x - 2y = 0$

For $m_1 = m_2 = -\frac{1}{2}$, (1) becomes $c \times 0 = 0$.

$\therefore$ c cannot be determined.

We go to $\frac{c^2}{2}\phi_n''(m) + c\,\phi'_{n-1}(m) + \varphi_{n-2}(m) = 0$

by case 3,

i.e., $-4c^2(1+6m)+4(2m-1)=0$ as

$$\varphi_{n-2}(m) = -4+8m.$$

Putting $m = -\frac{1}{2}$, $c^2(+2)-2=0$. $\therefore$ $c^2 = 1$. $c = \pm 1$

The two parallel asymptotes are

$$y = -\frac{x}{2} \pm 1$$

$$x + 2y = \pm 2$$

§ 5.1. Another method for finding asymptotes.

§ 5.2. Suppose the equation of the curve of the nth degree is put in the form $(ax + by + c)\,P_{n-1} + F_{n-1} = 0$ where P_{n-1} and F_{n-1} denote polynomials in x and y of the $(n - 1)^{th}$ degree. Any straight line parallel to $ax + by + c = 0$ cuts the curve in one point at infinity. To find the asymptote, we seek that member of this family parallel to $ax + by + c = 0$ which meets the curve in a second point at infinity. To find this, we allow x and y to tend to ∞ in the asymptotic direction $ax + by + c = 0$, i.e., $x : y : - b : a$.

∴ The asymptote is

$$ax + by + c + \underset{y = -\frac{a}{b}x \to \infty}{Lt}\left(\frac{F_{n-1}}{P_{n-1}}\right) = 0,$$

If this limit is finite, the asymptote we seek is found.

Examples.

Ex. 1. Find the asymptote of $x^3 + y^3 = 3axy$. (B.Sc. 1990)

(Vide Ex. 5 in Exercises 44.)

This equation can be written as

$$(x + y)(x^2 - xy + y^2) - 3axy + 0.$$

$\therefore$ The asymptotic direction is $x + y = 0$.

Hence the asymptote is

$$x + y + \underset{y = -x \to \infty}{Lt} \frac{-3axy}{x^2 - xy + y^2} = 0.$$

$$\text{i.e., } x + y + \underset{x \to \infty}{Lt} \frac{+3ax^2}{3x^2} = 0.$$

$$\text{i.e., } x + y + a = 0.$$

Ex. 2. Find the rectilinear asymptotes of

$$2x^4-5x^2y^2+3y^4+4x^3-6y^3+x^2+y^2-2xy+1 = 0.$$

Factorising the fourth degree terms, we have

$$(2x^2-3y^2)(x^2-y^2)+4x^3-6y^3+x^2+y^2-2xy+1=0.$$

The asymptotes are parallel to $\sqrt{2}\,x = \pm\sqrt{3}\,y$ and $x = \pm y$.

Hence

$$(\sqrt{2}\,x-\sqrt{3}\,y) + \underset{\sqrt{2}x=\sqrt{3}y\to\infty}{Lt} \frac{4x^3-6y^3+x^2+y^2-2xy+1}{(\sqrt{2}\,x+\sqrt{3}\,y)(x^2-y^2)} = 0.$$

i.e., $$\sqrt{2}\,x-\sqrt{3}\,y + \underset{y\to\infty}{Lt} \frac{3\sqrt{6}\,y^3-6y^3+5/2\,y^2-\sqrt{6}\,y^2+1}{2\sqrt{3}\,y\ \ 1/2\,y^2} = 0.$$

One asymptote is $\sqrt{2}\,x-\sqrt{3}\,y+(3\sqrt{2}-2\sqrt{3})=0$ (1)

Similarly

$$\sqrt{2}\,x+\sqrt{3}\,y + \underset{\sqrt{2}x=-\sqrt{3}y\to\infty}{Lt} \frac{4x^3-6y^3+x^2+y^2-2xy+1}{(\sqrt{2}\,x-\sqrt{3}\,y)(x^2-y^2)} = 0.$$

i.e., $$\sqrt{2}\,x+\sqrt{3}\,y + \underset{y\to\infty}{Lt} \frac{-3\sqrt{6}\,y^3-6y^3+5/2\,y^2+\sqrt{6}\,y^2+1}{-2\sqrt{3}\,y\cdot 1/2\,y^2} = 0.$$

i.e., $\sqrt{2}\,x+\sqrt{3}\,y+3\sqrt{2}+2\sqrt{3}=0$ - second asymptote.

Again, $$x-y+\underset{y=x\to\infty}{Lt} \frac{4x^3-6y^3+x^2+y^2-2xy+1}{(2x^2-3y^2)(x+y)} = 0$$

i.e., $x - y + \underset{x \to \infty}{Lt} \dfrac{-2x^3 - x^2 + 1}{-x^2(2x)} = 0$

i.e., $x - y + 1 = 0$ (3) - third asymptote.

$x + y + \underset{y = -x \to \infty}{Lt} \dfrac{4x^3 - 6y^3 + y^2 - 2xy + 1}{(2x^2 - 3y^2)(x - y)} = 0$

i.e., $x + y + \underset{x \to \infty}{Lt} \dfrac{10x^3 + 3x^2 + 1}{-x^2 \cdot 2x} = 0$

i.e., $x + y - 5 = 0$ (4) fourth asymptote.

§ 5.3. If the equation of the curve takes the form $(ax + by + c)\ F_{n-1} + F_{n-2} = 0$, the line $ax + by + c = 0$ cuts the curve in two points at infinity as the n^{th} and $(n-1)^{th}$ degree terms will be absent when we put $ax + by + c = 0$. Hence $ax + by + c = 0$ will be in general an asymptote. We use the qualifying word in general because there is an exception. Suppose F_{n-1} takes the form $(ax + by + c)\ P_{n-2}$ containing a factor $ax + by + c$, then there will be a pair of asymptotes parallel to $ax + by + c = 0$.

The curve can be then thrown into the form $(ax + by + c)^2\ P_{n-2} + F_{n-2} = 0$ and the parallel asymptotes are $ax + by + c = \pm \left(-Lt\ \dfrac{F_{n-2}}{P_{n-2}}\right)^{1/2}$ when x and $y \to \infty$ in the ratio

$$\frac{x}{y} = -\frac{b}{a}$$

If the curve can be written in the form

$$(ax + by)^2\ P_{n-2} + (ax + by)\ F_{n-2} + f_{n-2} = 0,$$

the parallel asymptotes are given by

$$(ax + by)^2 + (ax + by)\ Lt\ \frac{F_{n-2}}{P_{n-2}} + Lt\ \frac{f_{n-2}}{P_{n-2}} = 0$$

where x and $y \to \infty$ in the asymptotic direction $y = -\frac{bx}{a}$. Thus the parallel asymptotes are $ax + by = \alpha$ and $ax + by = \beta$, where α and β are the roots of $t^2 + t\, Lt \frac{f_{n-2}}{P_{n-2}} + Lt \frac{f_{n-2}}{P_{n-2}} = 0$

Examples

Ex. 1. Find the asymptotes of

$(x+y)^2(x+2y+2) = x+9y-2.$ (B.Sc. 1988)

Obviously two asymptotes are parallel to $x+y=0$; their equation are

$$(x+y)^2 = \underset{y=-x\to\infty}{Lt} \frac{x+9y-2}{x+2y+2} = \underset{x\to\infty}{Lt} \frac{-8x-2}{-x+2} = 8.$$

Hence they are $x+y = \pm 2\sqrt{2}$ (1)

The third asymptote is parallel to $x+2y=0$; its equation is (2)

$$x+2y+2 = \underset{x=-2y\to\infty}{Lt} \frac{x+9y-2}{(x+y)^2} = 0$$

Hence the asymptotes are

$x+y = \pm 2\sqrt{2}$ and $x+2y+2=0$

Ex. 2. Find the asymptotes of

$$(x-y)^2(x-2y)(x-3y) - 2a(x^3 - y^3) - 2a^2(x+y)(x-2y) = 0.$$

Obviously two asymptotes are parallel to $x - y = 0$.

Their equations are

$$(x-y)^2 - 2a(x-y) \underset{y=x\to\infty}{Lt} \frac{x^2+xy+y^2}{(x-2y)(x-3y)}$$

$$- 2a^2 \underset{y=x\to\infty}{Lt} \frac{(x+y)}{x-3y} = 0$$

i.e., $(x-y)^2 - 2a(x-y)\dfrac{3}{2} - 2a^2 . (-1) = 0$

i.e., $(x-y)^2 - 3a(x-y) + 2a^2 = 0$

$\therefore$ $x - y - a = 0$ and $x - y - 2a = 0$ are the two parallel asymptotes (1)

To find the asymptote parallel to $x - 2y$, we write the curve in the form

$$x - 2y - 2a \underset{x=2y\to\infty}{Lt} \frac{x^3 - y^3}{(x-y)^2 (x-3y)}$$

$$- 2a^2 \underset{x=2y\to\infty}{Lt} \frac{(x+y)(x-2y)}{(x-y)^2 (x-3y)} = 0$$

i.e., $x-2y-2a \underset{x=2y\to\infty}{Lt} \dfrac{x^2+xy+y^2}{(x-y)(x-3y)} = 0$

i.e., $x - 2y + 14a = 0$ (2)

The asymptote parallel to $x - 3y = 0$ is

$$x-3y+ \underset{x=3y\to\infty}{Lt} \frac{-2a(x^3-y^3)}{(x-y)^2 (x-2y)} + \underset{x=3y\to\infty}{Lt} \frac{-2a^2 (x+y)(x-2y)}{(x-y)^2 (x-2y)} = 0$$

i.e., $x - 3y - 13a = 0$ (3)

§ 6. Asymptotes by inspection. In certain cases, we can write down the equations of the asymptotes without any calculation. If the equation of the curve can be put in the form $F_n + F_{n-2} = 0$, where F_n breaks up into linear factors, so that no two of them represent parallel straight lines, then $F_n = 0$ represent the asymptotes.

Example.

Find the asymptotes of

$$(x+y)(x-y)(x-2y-4) = 3x + 7y - 6. \quad \text{(B.Sc. 1990)}$$

The asymptotes are obviously $x + y = 0$, $x - y = 0$ and

$$x - 2y - 4 = 0.$$

§ 7. Intersections of a curve with its asymptotes.

Let a curve of the n^{th} degree be of the form $F_n + F_{n-2} = 0$. We have already seen that the asymptotes of the curve are given by $F_n = 0$. Hence the n asymptotes meet the curve in points lying on the curve $F_{n-2} = 0$. Each asymptote, by definition, cuts the curve in the two coincident points at infinity and therefore, in $(n-2)$ other points. Hence these $n(n-2)$ points lie on a certain curve of degree $n-2$.

As particular cases,

(1) the asymptotes of a cubic cut the curve again in three points lying on a straight line.

(2) the asymptotes of a quartic curve cut the curve again in eight points lying on a conic.

Examples.

Ex. 1. Show that the asymptotes of the cubic

$$x^2y - xy^2 + xy + y^2 + x - y = 0$$

cut the curve again in three points which lie on the line $x + y = 0$

(B.Sc. 1988)

The asymptotes parallel to the axes are got by equating the coefficients of x^2 and y^2 to zero, i.e., $y = 0$ and $x = 1$.

The cubic can be written as $xy(x-y) + xy + y^2 + x - y = 0$. The third asymptote is parallel to $x-y = 0$. Its equation is

$$x - y + \underset{y=x\to\infty}{Lt} \frac{xy + y^2 + x - y}{xy} = 0$$

$$\text{i.e., } x - y + 2 = 0.$$

The combined equation of the asymptotes is

$$y(x-1)(x-y+2) = 0$$

i.e., $$A \equiv x^2y - xy^2 + xy + y^2 - 2y = 0.$$

The curve is $C \equiv x^2y - xy^2 + xy + y^2 + x - y = 0.$

The three points of intersection of A and C lie on $C - A = 0$.

i.e., $x + y = 0$.

Ex. 2. Determine the asymptotes of the curve $4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$ and show that they pass through the points of intersection of the curve with the ellipse $x^2 + 4y^2 = 4$. (B.Sc. 1988)

$(4x^2 - y^2)(x^2 - 4y^2) - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0.$

The asymptotes are parallel to $2x \pm y = 0$ and $x = \pm 2y$.

(1) The asymptote parallel to $2x + y = 0$ is

$$2x + y + \underset{y = -2x \pm \infty}{Lt} \frac{-4x(4y^2 - x^2) + 2(x^2 - 2)}{(2x - y)(x^2 - 4y^2)} = 0.$$

i.e., $2x + y + 1 = 0$ (1)

(2) The asymptote parallel to $2x - y = 0$ is

$$2x - y + \underset{y = 2x \pm \infty}{Lt} \frac{-4x(4y^2 - x^2) + 2(x^2 - 2)}{(2x + y)(x^2 - 4y^2)} = 0.$$

i.e., $2x - y + 1 = 0$ (2)

(2)

(3) The asymptote parallel to $x - 2y = 0$ is

$$\text{i.e., } x - 2y + \underset{x=2y\to\pm\infty}{Lt} \frac{-4x\,(4y^2 - x^2) + 2\,(x^2 - 2)}{(x+2y)\,(4x^2 - y^2)} = 0,$$

i.e., $x - 2y = 0$ (3)

[or $x-2y=0$ makes the 4th and 3rd degree terms vanish and hence is an asymptote].

(4) Similarly the other asymptote is $x + 2y = 0$ (4)

∴ The combined equation of the four asymptotes is

$$A \equiv (2x + y + 1)\,(2x - y + 1)\,(x - 2y)\,(x + 2y)$$

$$\equiv (4x^2 - y^2)\,(x^2 - 4y^2) + 4x\,(4y^2 - x^2) + (x^2 - 4\,y^2) = 0.$$

The quartic equation

$C \equiv (4x^2 - y^2)(x^2 - 4y^2) + 4x(4y^2 - x^2) + 2(x^2 - 2) = 0.$

Hence the eight points of intersection of the quartic and the asymptotes lie on $C - A = 0$, i.e., $x^2 + 4y^2 - 4 = 0$, an ellipse.

Ex. 3. Find the equation of a cubic which has the same

the cubic

Hence the ... asymptotes lie on C − A = 0, i.e., $x^2 + 4y^2 - 4 = 0$, ...

Ex. 3. Find the equation of a cubic which has the same asymptotes as the cubic $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which touches the axis of y at the origin and goes through the point (3,2)

As the cubic is of the form $F_3 + F_1 = 0$, the asymptotes are given by $x^3 - 6x^2y + 11xy^3 - 6y^3 = 0$. (Vide §6) The equation of a cubic which has the same asymptotes is $x^3 - 6x^2 + 11xy^2 - 6y^3 + ax + by + c = 0$. As the cubic touches the y-axis at the origin, it passes through the origin ; hence $c = 0$. Besides, the tangent at the origin is $x = 0$; hence the lowest degree terms must contain only x. Therefore $b = 0$. So the cubic takes the form $x^3 - 6x^2y + 11xy^2 - 6y^3 + ax = 0$. As (3,2) lies on the cubic, $a = -1$. Hence the cubic is $x^3 - 6x^2y + 11xy^2 - 6y^3 = x$.

... equation of quartic which has $y = \pm x \pm 1$ as

$a = -1$. Hence the cubic is x ...

Ex. 4. Find the equation of quartic which has $y = \pm x \pm 1$ as asymptotes, which cuts the x - axis in four contiguous points at the origin and the y-axis in three points (other than the origin) for which the product of the ordinates is -1.

The equation of the quartic will be of the form

$$(y - x + 1)(y - x - 1)(y + x + 1)(y + x - 1)$$
$$+ \lambda (y - x)(y + x) + ax + by + c = 0 \text{ (by §6)}$$

i.e., $(x^2 - y^2)^2 - 2(x^2 + y^2) + 1 + \lambda (y^2 - x^2) + ax + by + c = 0.$

As the curve passes through the origin $c + 1 = 0$ (1) Putting $y = 0$, we get $x^4 - 2x^2 - \lambda x^2 + a = 0$. This must be $x^4 = 0$ as the x- axis is given to cut the curve in four coincident points at the origin. Hence $\lambda = -2$ and $a = 0$ (2)

THANK YOU

B. Sc., I YEAR- I SEMESTER

calculus

COURSE CODE: 7BMA1C1

UNIT-3

TEXTBOOK: S.NARAYANAN AND T.K. MANICKAVACHAGAMPILLAY,S.VISWANATHAN

CHAPTER-1[11-12]

PART-10

R.RAJALAKSHMI [G.L],GACW,RAMANATHAPURAM

6. Show that $\int_0^{\pi/2} \frac{dx}{1 + a^2 \cos^2 x + b^2 \sin^2 x}$

$$= \frac{\pi}{2} \frac{1}{\sqrt{(1 + a^2)(1 + b^2)}}$$

7. Show that

$$\int \frac{dx}{1 - \sin^4 x} = \frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1} (\sqrt{2} \tan x)$$

§ 11. Properties of definite integrals.

1. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$. This is obvious from the definition of a definite integral.

2. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where c is some value of x between a and b.

Let $\int f(x)\, dx = F(x)$

Then $\int_a^b f(x)\, dx = F(b) - F(a)$.

The R.H.S. $= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$. Hence the result.

3. $\int_{-a}^{+a} f(x)\, dx = 2\int_0^a f(x)\, dx$, if $f(x)$ is an even function of x.

If $f(x)$ is even, $f(x) = f(-x)$.

$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx$ by (2)

$= \int_{-a}^{0} f(-x)\, dx + \int_0^a f(x)\, dx$

$= -\int_a^0 f(y)\, dy + \int_0^a f(x)\, dx$ (by putting $y = -x$ in the first integral)

$= \int_0^a f(y)\, dy + \int_0^a f(x)\, dx = 2\int_0^a f(x)\, dx$

as in a definite integral we can replace the variable y by x.

4. If $f(x)$ is an odd function of x, $\int_{-a}^{a} f(x)\, dx = 0$.

If $f(x)$ is odd, $f(x) = -f(-x)$.

$$\therefore \int_{-a}^{a} f(x)\, ax = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx$$

$$= -\int_{-a}^{0} f(-x)\, dx + \int_0^a f(x)\, dx$$

$$= +\int_a^0 f(y)\, dy + \int_0^a f(x)\, dx$$

$$= -\int_0^a f(x)\, dx + \int_0^a f(x)\, dx$$

$$= 0.$$

5. $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$.

In $\int_0^a f(a - x)\, dx$, put $a - x = y$

R.H.S. $= -\int_a^0 f(y)\, dy = \int_0^a f(y)\, dy = \int_0^a f(x)\, dx$.

This result is very useful in evaluating many integrals.

Examples.

Ex. 1. Prove that $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$

Let $f(x) = \sin^n x$. Here $a = \frac{\pi}{2}$.

$$\therefore f(a - x) = \sin^n\left(\frac{\pi}{2} - x\right) = \cos^n x.$$

By § 11.5 the result follows.

Ex. 2. $\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + \cos x^{3/2}}\,dx = \frac{\pi}{4}.$

Let I be the value of this integral and $f(x)$ denote the integrand $\frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}$.

$$\therefore I = \int_0^{\frac{\pi}{2}} f(x)\,dx.$$

$f(a - x) = \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}}$ as $a = \frac{\pi}{2}$ here,

Also $\quad I = \int_0^{\frac{\pi}{2}} f(a - x)\,dx.$

Adding (1) and (2),

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\,dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

Hence $I = \frac{\pi}{4}$.

Ex. 3. $\int_0^{\pi/4} \log(1 + \tan\theta\,d\theta = \frac{\pi}{8}\log 2.$

Let $f(\theta) = \log(1 + \tan\theta)$. Here $a = \frac{\pi}{4}$.

$$\therefore f\left(\frac{\pi}{4} - \theta\right) = \log\left\{1 + \tan\left(\frac{\pi}{4} - \theta\right)\right\}$$

$$= \log\left\{1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right\} = \log\frac{2}{1 + \tan\theta}$$

$$I = \int_0^{\pi/4} \log(1 + \tan\theta)\,d\theta$$

and $\quad I = \int_0^{\pi/4} \log\frac{2}{1 + \tan\theta}\,d\theta$ by § 11.5

Adding, $2I = \int_0^{\pi/4} \log 2\,d\theta = \log 2\,[\theta]_0^{\pi/4}$

$$= \frac{\pi}{4}\log 2.$$

Hence the result.

Ex. 4. $\int_0^{\pi} \theta\sin^3\theta\,d\theta = \frac{2\pi}{3}.$

$f(\theta) = \theta\sin^3\theta$. Here $a = \pi$.

$\therefore \quad f(a - \theta) = (\pi - \theta)\sin^3\theta.$

Hence $I = \int_0^{\pi} \theta\sin^3\theta\,d\theta$ and $I = \int_0^{\pi} (\pi - \theta)\sin^3\theta\,d\theta$ by §11.5

Adding, $2I = \pi\int_0^{\pi} \sin^3\theta\,d\theta$

$= \pi\int \sin^2\theta(-dy)$ putting $\cos\theta = y$; $-\sin\theta\,d\theta = dy$

$$= -\pi \int_1^{-1} (1 - y^2)\, dy = -\pi \left[y - \frac{y^3}{3}\right]_1^{-1}$$

$$= -\pi \left[-1 + \frac{1}{3} - 1 + \frac{1}{3}\right] = \frac{4\pi}{3}.$$

Hence $I = \frac{2\pi}{3}$.

Ex. 5. $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$ if $f(2a - x) = f(x)$.

and $= 0$ if $f(2a - x) = -f(x)$.

$$\int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_a^{2a} f(x)\, dx$$

In the second integral, put $2a - x = y$; $dx = -dy$

When $x = a$, $y = a$ and $x = 2a$, $y = 0$.

Hence $\int_a^{2a} f(x)\, dx = -\int_a^{0} f(2a - y)\, dy = \int_0^{a} f(2a - y)\, dy$

$$= \int_0^{a} f(2a - x)\, dx.$$

$\therefore \int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx$ from (1).

If $f(2a - x) = f(x)$, $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$.

If $f(2a - x) = -f(x)$, $2\int_0^{a} f(x)\, dx = 0$.

Cor. $\int_0^{\pi} f(\sin x)\, dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\, dx$.

Ex. 6. Evaluate $I = \int_0^{\frac{\pi}{2}} \log \sin x\, dx$

$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x\, dx$ (by §11.5).

Hence $2I = \int_0^{\frac{\pi}{2}} \log \sin x\, dx + \int_0^{\frac{\pi}{2}} \log \cos x\, dx$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \frac{\pi}{2}\log 2.$$

Put $2x = z$; $dx = \frac{1}{2} dz$; then

$$\int_0^{\frac{\pi}{2}} \log \sin 2x\, dx = \frac{1}{2}\int_0^{\pi} \log \sin z\, dz = \frac{1}{2}\int_0^{\pi} \log \sin x\, dx$$

$$= \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx \text{ (by 5 Cor.)}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x\, dx.$$

Thus, $2I = I - \frac{\pi}{2}\log 2$.

25. If $f(x) = f(a + x)$, show that

(i) $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx.$

(ii) $\int_a^{na} f(x)\,dx = (n-1)\int_0^{a} f(x)\,dx.$

§ 12. Integration by parts.

If u and v are functions of x,

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ by the product rule.

Integraing both sides with respect to x

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$\therefore\ uv = \int u\,dv + \int v\,du.$

Hence $\int u\,dv = uv - \int v\,du.$

Note : - The success of this method depends on the proper choice of u and v ; the auxiliary integral $\int v\,du$ must be easier to integrate than the given integral.

Examples.

Ex. 1. $\int x e^x\,dx$

Writing $dv = e^x\,dx$ and $u = x$, $v = \int e^x\,dx = e^x$.

$\therefore \int x e^x\,dx = \int x d(e^x) = \int u\,dv = uv - \int v\,du$

$= xe^x - \int e^x\,dx = xe^x - e^x = e^x(x-1).$

Ex. 2. $\int x \sin 2x\,dx.$

Here $dv = \sin 2x\,dx$, i.e., $v = \int \sin 2x\,dx = \frac{-\cos 2x}{2}$ and $u = x$.

$\therefore \int x\sin 2x\,dx = \int x d\left(\frac{-\cos 2x}{2}\right) = \int u\,dv = uv - \int v\,du$

$$= -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}$$

Ex. 3. $\int x^n \log x\,dx$. Put $u = \log x$ and $dv = x^n\,dx$.

i.e., $v = \int x^n\,dx = \frac{x^{n+1}}{n+1}$

$$\therefore \int x^n \log x\,dx = \int \log x d\left(\frac{x^{n+1}}{n+1}\right)$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx.$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n\,dx.$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$$

Ex. 4. $\int \sin^{-1} x\,dx$

Put $u = \sin^{-1} x$ and $dv = dx$, i.e., $v = x$

$$\int \sin^{-1} x\,dx = \int u\,dv = uv - \int u\,du = x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$= x\sin^{-1}x - \int \sin\theta\,d\theta$ on putting $x = \sin\theta$.

$= x\sin^{-1}x + \cos\theta = x\sin^{-1}x + \sqrt{1-x^2}.$

Ex. 5. $\int \tan^{-1} x\,dx$. [Here $u = \tan^{-1}x$ and $v = x$]

$$= x\tan^{-1}x - \int \frac{x\,dx}{1+x^2} = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

Ex. 6. $\int x^2 \tan^{-1} x\,dx$. [Here $u = \tan^{-1}x$; $dv = x^2\,dx$.

$\therefore\ v = x^3/3.$]

$$= \int \tan^{-1}x\, d\left(\frac{x^3}{3}\right) = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left\{\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\right\}.$$

Ex. 7. $\int (\log x)^2\,dx.$ (B.Sc. 1994)

Here $u = (\log x)^2$ and $v = x$.

$$\therefore \int (\log x)^2\,dx = x(\log x)^2 - \int x \,.\, 2\log x \,.\, \frac{1}{x}\,dx$$

$$= x(\log x)^2 - 2\int \log x\,dx$$

$$= x(\log x)^2 - 2\left(x\log x - \int x \,.\, \frac{1}{x}\,dx\right)$$

$$= x(\log x)^2 - 2x\log x + 2x.$$

Ex. 8. $\int \sqrt{a^2+x^2}\,dx.$

[Here $u = \sqrt{a^2+x^2}$ and $v = x$.]

$$\text{Integral} = x\sqrt{a^2+x^2} - \int \frac{x^2\,dx}{\sqrt{a^2+x^2}}$$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2+x^2}}$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2\sinh^{-1}\frac{x}{a}.$$

Transposting, we get

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2\sinh^{-1}\frac{x}{a}\right)$$

Ex. 9. $\int \frac{x+\sin x}{1+\cos x}\,dx.$

$$I = \int \frac{x\,dx}{1+\cos x} + \int \frac{\sin x\,dx}{1+\cos x}$$

$$= \int \frac{x\,dx}{2\cos^2\left(\frac{x}{2}\right)} + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \int x\,d\left(\tan\frac{x}{2}\right) + \int \tan\frac{x}{2}\,dx.$$

$$= x\tan\frac{x}{2} - \int\tan\frac{x}{2}\,dx + \int\tan\frac{x}{2}\,dx = x\tan\frac{x}{2}.$$

Ex. 10. $\int e^x\,\frac{x+1}{(x+2)^2}\,dx$

$$= \int e^x\,\frac{x+2-1}{(x+2)^2}\,dx = \int\frac{e^x}{x+2}\,dx - \int\frac{e^x}{(x+2)^2}\,dx$$

$$= \int\frac{1}{x+2}\,d(e^x) - \int\frac{e^x}{(x+2)^2}\,dx$$

$$= \int\frac{e^x}{x+2} + \int\frac{e^x}{(x+2)^2}\,dx - \int\frac{e^x\,dx}{(x+2)^2} = \frac{e^x}{x+2}.$$

Ex. 11. $\int e^x(\sin x + \cos x)\,dx = \int e^x\sin x\,dx + \int e^x\cos x\,dx.$

$$= \int \sin x\,d(e^x) + \int e^x\cos x\,dx$$

$$= \sin x\,e^x - \int e^x\cos x\,dx + \int e^x\cos x\,dx = \sin x\,e^x.$$

Exercies 17.

Integrate

§ 1.3. Reduction formulae.

§ 13.1. $I_n = \int x^n e^{ax}\,dx$, where n is a positive integer.

Here $dv = e^{ax}\,dx$, *i. e.*, $v = \int e^{ax}\,dx = \frac{e^{ax}}{a}$ and $u = x^n$.

$$\therefore\ I_n = \int x^n\, d\left(\frac{e^{ax}}{a}\right) = \frac{e^{ax}}{x} x^n - \frac{n}{a}\int e^{ax} x^{n-1}\,dx$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}.$$

The auxiliary integral is of the same type as the given integral but with index n reduced by 1. Such a formula is called a reduction formula and by successive applications, we can evaluate I_n

The ultimate integral is obviously $\int e^{ax}\,dx = \frac{e^{ax}}{a}$.

§ 13.2. $I_n = \int x^n \cos ax\,dx$ (n a positive integer).

$$I_n = \int x^n \cos ax\,dx = \int x^n d\left(\frac{\sin ax}{a}\right) \quad \left[\text{Here } u = x^n \text{ and } v = \frac{\sin ax}{a}\right]$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax\,dx$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} d\left(\frac{-\cos ax}{a}\right)$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\,dx$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2} I_{n-2}$$

The ultimate integral is either $\int x \cos ax\,dx$ or $\int \cos ax\,dx$ accordimg as n is odd or even.

(i) $\int x \cos ax\,dx = \int x d\left(\frac{\sin ax}{a}\right) = \frac{x \sin ax}{a} - \frac{1}{a}\int \sin ax\,dx$

$$= \frac{x \sin ax}{a} + \frac{1}{a^2}\cos ax.$$

(ii) $\int \cos ax\,dx = \frac{\sin ax}{a}$.

Exercises 18.

9. Evaluate $\int_0$

§ 13.3. $I_n = \int \sin^n x\,dx$ (n being a positive integer)

$I_n = \int \sin^{n-1} x \sin x\,dx = \int \sin^{n-1} x\,d(-\cos x)$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\,I_{n-2} - (n-1)\,I_n.$

$\therefore\; n\,I_n = -\sin^{n-1} x \cos x + (n-1)\,I_{n-2}.$

The ultimate integral is $\int \sin x\,dx$ or $\int dx$ according as n is odd or even, i.e., $-\cos x$ or x.

Corollary

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\sin^{n-1}x \cos x}{n}\right]_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$= \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$ as the first term vanishes at both limits.

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4} x\,dx.$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\,\cdots\cdots$$

If n is even, the ultimate integral is $\int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2}.$

If n is odd, the ultimate integral is

$$\int \sin x\,dx = (-\cos x)_0^{\pi/2} = 1.$$

CALCULUS

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\,\frac{n-1}{n-2}\cdots\frac{1}{2}\,\frac{\pi}{2}\quad \text{when } n \text{ is even and}$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{2}{3},\quad \text{when } n \text{ is odd.}$$

Examples.

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\frac{n-1}{n-2}\cdots\frac{1}{2}\frac{\pi}{2} \text{ when } n \text{ is even and}$$

$$= \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}, \text{ when } n \text{ is odd.}$$

Examples.

Ex. 1. $\int_0^{\pi/2} \sin^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\frac{\pi}{2} = \frac{5\pi}{32}.$

Ex. 2. $\int_0^{\pi/2} \sin^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{48}{105} = \frac{16}{35}$

Ex. 3. In $\int \sin^n x\,dx$, if n be an odd positive integer, we can directly integrate without using the reduction formula. For instance let us find $\int \sin^5 x\,dx$.

Put $y = \cos x$; $dy = -\sin x\,dx$

$$\int \sin^5 x\,dx = -\int \sin^4 x\,dy = -\int (1-y^2)^2\,dy$$

$$= -\int (1 - 2y^2 + y^4)\,dy$$

$$= -y + \frac{2y^3}{3} - \frac{y^5}{5} = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}$$

Ex. 4. Evaluate $\int_0^{\pi/2} x(1-x^2)^{1/2}\,dx$

Put $x = \sin\theta$; $dx = \cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \frac{\pi}{2}$

The integral becomes

$$\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta = \int \cos^2\theta\,d(-\cos\theta) = \left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{3}$$

§ 13. 4. $I_n = \int \cos^n x\,dx$ (n being a positive integer).

$$I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$$

$$= \int \cos^{n-1} x\,d(\sin x)$$

$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

$$\therefore\ n\,I_n = \cos^{n-1} x \sin x + (n-1)\,I_{n-2}$$

The ultimate integral is $\int \cos x\,dx$ or $\int dx$, i.e., $\sin x$ or x according as n is odd or even.

Corollary.

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{\cos^{n-1} x \sin x}{n}\right)_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

$= \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$ as the first term vanishes at both limits.

$$= \frac{n-1}{n}\frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx$$

$$= \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots$$

The ultimate integral is

$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$ when n is odd.

The ultimate integral is

$\int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$, when n is even.

For example, take

(1) $\int \sin^6 x \cos^3 x\, dx$. Put $y = \sin x$; $dy = \cos x\, dx$

$$\int \sin^6 x \cos^3 x\, dx = \int y^6 (1 - y^2)\, dy = \frac{y^7}{7} - \frac{y^9}{9}$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9}.$$

(2) $\int \sin^9 x \cos^5 x\, dx$. Put $\sin x = y$; $\cos x\, dx = dy$.

$$\int \sin^9 x \cos^5 x\, dx = \int y^9 (1 - 2y^2 + y^4)\, dy$$

$$= \frac{y^{10}}{10} - \frac{y^{12}}{6} + \frac{y^{14}}{14} = \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{6} + \frac{\sin^{14} x}{14}$$

Case (ii). Let both m and n be even +ve integers. Let $n < m$. Applying (a), the ultimate integral is

$$I_{m,0} = \int \sin^m x\, dx$$

which has been discussed in §13.3.

Corollary.

$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ (m,n being positive integers).

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$

$$= \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$ as the first term vanishes at both limits

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \int_0^{\pi/2} \sin^m x \cos^{n-4} x\, dx$$

$$\frac{n-1}{m+n} \frac{n-3}{m+n-2} \frac{n-5}{m+n-4} \ldots I_{m,1} \text{ or } I_{m,0}$$

according as n is odd or even.

(i) If n is odd, $I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x\, dx$

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}.$$

When n is odd,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \ldots \frac{2}{m+3} \frac{1}{m+1}.$$

(ii) If n is even,

$$I_{m,0} = \int_0^{\pi/2} \sin^m x\, dx = \frac{m-1}{m} \frac{m-3}{m-2} \ldots \frac{1}{2} \frac{\pi}{2}$$ by §13.3. Cor.

when m is even

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \frac{n-1}{m+n} \frac{n-3}{m+n-2} \cdots \frac{1}{m+1} \frac{m-1}{m} \frac{m-3}{m-2} \cdots \frac{1}{2} \frac{\pi}{2}$$

Examples.

Ex. 1. $\int_0^{\pi/2} \sin^6 x \cos^5 x\, dx = \frac{4}{11} \cdot \frac{2}{9} \cdot \frac{1}{7}$ by (i)

$$= \frac{8}{693}.$$

Ex. 2. $\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx = \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} = \frac{3\pi}{512}.$

§ 13.6. $I_n = \int \tan^n x\, dx$ (n being a positive integer)

(B.Sc. 1994)

$$I_n = \int \tan^{n-2} x \tan^2 x \, dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2} x \, d(\tan x) - \int \tan^{n-2} x \, dx$$

$$= \frac{\tan^{n-1}}{n-1} - I_{n-2}$$

(i) When n is even, the ultimate integral is $\int dx = x$

(ii) When n is odd, the ultimate integral is $\int \tan x \, dx = \log \sec x$ (Vide § 6.5, Ex. 3)

Examples.

Ex. 1. $\int \tan^4 x \, dx = \dfrac{\tan^3 x}{3} - \int \tan^2 x \, dx$ by putting $n = 4$ in the formula for I_n

$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) \, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x$$

Ex. 2. $\int_0^{\pi/4} \tan^3 x \, dx = \left[\dfrac{\tan^2 x}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$ by putting $n = 3$

$$= \frac{1}{2} + [\log \cos x]_0^{\pi/4} = \frac{1}{2} + \log \frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \log 2)$$

§ 13. 7. $I_n = \cot^n x \, dx$ (n being a positive integer).

$$\int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx$$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= \int \cot^{n-2} x \, d(-\cot x) - \int \cot^{n-2} x \, dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

The ultimate integral is $\int dx$ or $\int \cot x \, dx$, i.e., x or $\log \sin x$ according as n is even or odd.

§ 13. 8. $I_n = \int \sec^n x \, dx$ (n being a positive integer).

$$\int \sec^n x \, dx = \int \sec^{n-2} x \, d(\tan x)$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

(i) If n be an odd integer, the ultimate integral is $\int \sec x \, dx = \log (\tan x + \sec x)$. (Vide § 6.5 Ex. 5)

(ii) If n be an even integer, the ultimate integral is $\int dx = x$.

Examples.

Ex.1. $I = \int \sec^3 x \, dx = \int \sec x \, d(\tan x)$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - I + \log (\sec x + \tan x)$$

$$\therefore 2I = \sec x \tan x + \log (\sec x + \tan x).$$

Ex. 2. $\int \sec^6 x \, dx = \int \sec^4 x \, d(\tan x) = \int (1 + t^2)^2 \, dt$ (where $t = \tan x$) ;

$= \int (1 + 2t^2 + t^4)\, dt = t + \frac{2t^3}{3} + \frac{t^5}{5}$

$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5}$

§ 13. 9. $I_n = \int \operatorname{cosec}^n x\, dx$ (n being a positive integer).

$I_n = \int \operatorname{cosec}^n x\, dx = -\int \operatorname{cosec}^{n-2} x\, d(\cot x)$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \cot^2 x\, dx$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \cdot (\operatorname{cosec}^2 x - 1)\, dx$

$= -\operatorname{cosec}^{n-2} \cot x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

(i) If n be an odd integer, the ultimate integral is $\int \operatorname{cosec} x\, dx = -\log(\operatorname{cosec} x + \cot x)$ [Vide § 6.5. Ex. 6]

(ii) If n be an even integer, ultimate integral is $\int dx = x$.

Examples.

Ex. 1. $\int \operatorname{cosec}^4 x\, dx = -\int \operatorname{cosec}^2 x\, d(\cot x)$

$= -\int (1 + y^2)\, dy$, where $y = \cot x$

$= -y - \frac{y^3}{3} = -\cot x - \frac{\cot^3 x}{3}$

Ex. 2. $\int \operatorname{cosec}^5 x\, dx$

Putting $n = 5$ in the above formula for I_n

$\int \operatorname{cosec}^5 x\, dx = -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4} \int \operatorname{cosec}^3 x\, dx$

$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8} \operatorname{cosec} x \cot x$

$- \frac{3}{8} \log(\operatorname{cosec} x + \cot x)$

§ 13. 10. $I_{m,n} = \int x^m (\log x)^n\, dx$ (where m and n are positive integers)

Hence or otherwise evaluate $\int x^4 (\log x)^3\, dx$.

$I_{m,n} = \int (\log x)^n\, d\left(\frac{x^{m+1}}{m+1}\right)$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1}\, dx$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$.

The ultimate integral is $I_{m,0} = \int x^m\, dx = \frac{x^{m+1}}{m+1}$.

$\int (\log x)^3 x^4\, dx = \int (\log x)^3\, d\left(\frac{x^5}{5}\right)$

$= \frac{x^5}{5} (\log x)^3 - \frac{3}{5} \int (\log x)^2 x^4\, dx$

$= \frac{x^5}{5} (\log x)^3 - \frac{3}{5} \int (\log x)^2\, d\left(\frac{x^5}{5}\right)$

$= \frac{x^5}{5} (\log x)^3 - \frac{3}{25} x^5 (\log x)^2 + \frac{6}{25} \int x^4 (\log x)\, dx$

$= \frac{x^5}{5} (\log x)^3 - \frac{3}{25} x^5 (\log x)^2 + \frac{6}{25} \left\{\frac{x^5}{5} \log x - \frac{x^5}{25}\right\}$

$= x^5 \left\{\frac{1}{5} (\log x)^3 - \frac{3}{25} (\log x)^2 + \frac{6}{125} \log x - \frac{6}{625}\right\}$.

$I_{2n+1} = \int_0$

§ 14. $\int e^{ax} \cos bx \, dx$, a and b are constants.

Let $C = \int e^{ax} \cos bx \, dx$ and $S = \int e^{ax} \sin bx \, dx$.

$C + iS = \int e^{ax} (\cos bx + i \sin bx) \, dx$

$= \int e^{ax} e^{ibx} \, dx$ (by Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$)

$= \int e^{x(a+ib)} \, dx = \dfrac{e^{(a+ib)x}}{a + ib}$

$= e^{ax} \dfrac{(a - ib)(\cos bx + i \sin bx)}{a^2 + b^2}$

$C = \text{Real part of } \; e^{ax} \dfrac{(a - ib)(\cos bx + i \sin bx)}{a^2 + b^2}$

$= e^{ax} \dfrac{(a \cos bx + b \sin bx)}{a^2 + b^2}$

$S = \text{Imaginary part of } \; e^{ax} \dfrac{(a - ib)(\cos bx + i \sin bx)}{a^2 + b^2}$

$$= e^{ax}\,\frac{a \sin bx - b \cos bx}{a^2 + b^2}.$$

Examples.

Ex. 1. $\int e^{2x} \cos 3x\, dx.$

$C = \int e^{2x}\, 3x\, dx$ = Real part of $\int e^{2x} e^{3ix}\, dx$

= Real part of $\int e^{x(2+3i)}\, dx$

= Real part of $\dfrac{e^{x(2+3i)}}{2+3i}$

= Real part of $\dfrac{e^{2x}}{13}(2-3i)\,(\cos 3x + i \sin 3x)$

$= \dfrac{e^{2x}}{13}(2 \cos 3x + 3 \sin 3x).$

Ex. 2. $\int e^{-x} \sin^2 x\, dx = \int e^{-x}\, \dfrac{1-\cos 2x}{2}\, dx$

$= -\dfrac{e^{-x}}{2} - \dfrac{1}{2}\int e^{-x} \cos 2x\, dx$

$= -\dfrac{e^{-x}}{2} - \dfrac{1}{2} e^{-x}\, \dfrac{-\cos 2x + 2 \sin 2x}{5}$

by putting $a = -1$ and $b = 2$ in the above formula.

Ex. 3. $\int e^{ax} \cos mx \cos nx\, dx = \dfrac{1}{2}\int e^{ax}\, \{\cos (m+n)\, x + \cos (m-n)\, x\}\, dx$

$$= \frac{1}{2} e^{ax} \left\{ \frac{a \cos (m+n)\, x + (m+n) \sin (m+n)\, x}{a^2 + (m+n)^2} + \frac{a \cos (m-n)\, x + (m-n) \sin (m-n)\, x}{a^2 + (m-n)^2} \right.$$

es 20

Integrate

1. $e^x \sin 2x$
2. $e^{-3x} \sin \frac{x}{2}$.
3. $e^{4x} \cos 3x$.
4. $e^{m \cos^{-1} x}$.
5. $e^{ax} \sin (bx + c)$.
6. $e^x \cos^2 x$.
7. $e^{2x} \cos (3x + 4)$.
8. $e^x \sin 3x \cos 2x$.
9. $e^{2x} \cos 5x \cos 4x$.
10. $e^{-3x} \sin 3x \sin 2x$.
11. $a^x \sin x$.
12. $3^x \sin 2x$.

§ 15.1. Bernoulli's formula.

This formula is merely an extension of the formula of integration by parts.

Let dashes denote successive differentiation and suffixes denote successive integration:

e.g., $\dfrac{du}{dx} = u'$; $\dfrac{d^2u}{dx^2} = u''$ etc., and $\int v dx = v_1$;

$\int\int v\, (dx)^2 = v_2$, etc.

$\int u\, dv = uv - \int v\, du$ by § 12

$= uv - \int u'\, d\,(v_1)$ with the above notation by §12

$= uv - u'v_1 + \int v_1\, du'$

$= uv - u'v_1 + \int u''\, d\,(v_2)$

$= uv - u'v_1 + u''v_2 - \int v_2\, du''$ by §12

$= uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

The advantage of the above formula can be seen in examples where the tediousness of successive integration by parts is avoided.

THANK YOU

UNIT - IV

MULTIPLE INTEGRALS

1. Integration may be considered either as the inverse of differentiation or as a process of summation

Let $f(x)$ be a continuous function in the closed interval from $x=a$ to $y=b$. Hence the function is bounded in the interval. Let $b>a$. Divide the interval (a,b) into n sub-intervals $x_1-a, x_2-x_1, x_3-x_2 \ldots b-x_{n-1}$ where $a, x_1, x_2, x_3 \ldots x_{n-1}, b$ are in ascending order of magnitudes. Let ξ_1 be any point of the sub interval (x_{r-1}, x_r). Taking $a=x_0$ and $b=x_n$, consider the sum.

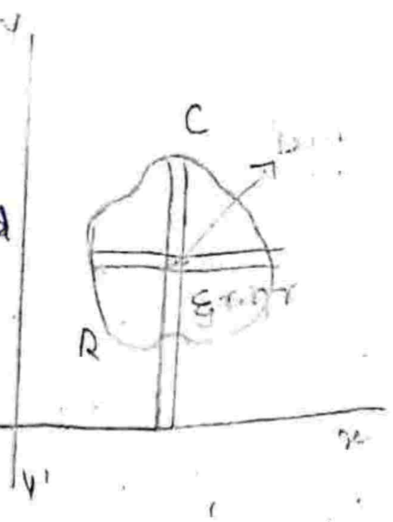
$$f(x_1)(x_1-x_0)+f(x_2)(x_2-x_1)\ldots+f(x_n)(x_n-x_{n-1})$$

This sum tends to a definite limit when the number n of the sub-intervals tends to infinity. The lenght of each sub interval tends to zero, as a a and b are finite. we have already seen that this limit is called definite integral of f(x) with respect to x from $x=a$ to $x=b$ and is written as $\int_a^b f(x)\,dx$. Even in the case of simple functions the evalution of an integral from this definition is not quite easy. So we evalute $\int_a^b f(x)\,dx$ from the following result:

$$\int_a^b f(x)\,dx = F(b)-F(a), \text{ where } \frac{d}{dx}F(x)=f(x)$$

Definition of the Double integral

The double integral is defined in a similar manner. Let $f(x,y)$ be a continuous and single valued function of x and y within a region R bounded by a closed curve C and



upon the boundary C. Let the region R be subdivided in any manner into n sub regions of areas $\Delta A_1, \Delta A_2 \ldots \Delta A_n$

Let (ξ_r, η_r) be any point in the sub region of area ΔA_r and consider the sum $\sum_{r=1}^{n} (\xi_r, \eta_r)\Delta A_r$.

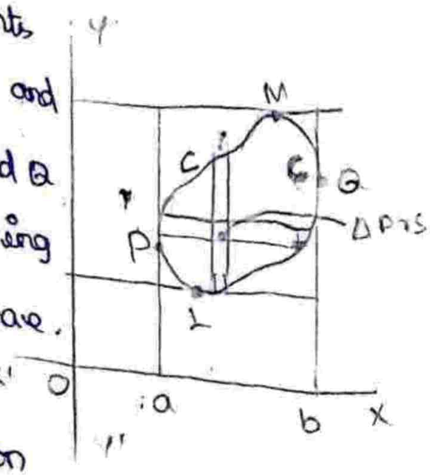
The limit of this sum as $n \to \infty$ and $\Delta A_r \to 0$ $(r = 1, 2, \ldots)$ is defined as the double integral of $f(x,y)$ over the region R.

Thus $\iint_R f(x,y)\,dA = \lim_{n \to \infty} \sum_{r=1}^{n} (\xi_r, \eta_r)\Delta A_r$

The region R is called the region of integration corresponding to interval of integration (a,b) in the case of the simple integral. This integral is sometimes written as $\iint_R f(x,y)\,dx\,dy$.

Evalution of the double integral

Let L and M be the points on curve C having the minimum and maximum ordinates and let P and Q be the points on the curve C having the minimum and maximum abscissae. Let $x = f_1(y)$ be the equation of LPM and $x = f_2(y)$ be the equation LQM. Divide the range (a, b) along the x axis into n equal parts and draw lines parallel to Y axis through the points of intersection. Divide the range (c, d) along the Y axis into m equal parts and draw lines parallel to x axis through the points of intersection. Then the region R is subdivided into sub-regions ΔR_{rs} of area ΔA_{rs}, where $\Delta A_{rs} = \Delta x_r \cdot \Delta y_s$



Let (ξ_{rs}, η_{rs}) be any point of ΔR_{rs}

The sum $\sum_{r=1}^{r=n} \sum_{s=1}^{s=m} f(\xi_{rs}, \eta_{rs}) \Delta A_{rs}$ can be written as

$$\sum_{r=1}^{r=n} \sum_{s=1}^{s=m} f(\xi_{rs}, \eta_{rs}) \Delta x_r \, \Delta y_s$$

This Summation Signifies that the terms can be Summed for r and s in any manner we like, we can take the Sum of the terms by rows and add all these Sums.

Hence the Sum (i) can be written as

$$\sum_{s=1}^{m} \Delta y_s \left[\sum_{r=1}^{n} f(\xi_{rs}, \eta_{rs}) \Delta x_r\right]$$

But $\lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_{rs}, \eta_{rs}) \Delta x_r = \int_{f_1(\eta_s)}^{f_2(\eta_s)} f(x, \eta_s) dx$

$\sum_{r=1}^{n} (\xi_{rs}, \eta_{rs}) \Delta x_r = \int_{f_1(\eta_s)}^{f_2(\eta_s)} f(x, \eta_s) dx + \epsilon_s$, where $\epsilon_s \to 0$ as $n\to\infty$

Also $\int_{f_1(\eta_s)}^{f_2(\eta_s)} f(x, \eta_s) dx$ is a function of η_s.

Let it be equal to $F(\eta_s)$

Then (i) becomes $\sum_{s=1}^{m} [F(\eta_s) + \epsilon_s] \Delta y_s$

$$= \sum_{s=1}^{m} F(\eta_s) \Delta y_s + \sum_{s=1}^{m} \epsilon_s \Delta y_s$$

$$\lim_{m\to\infty} \sum_{s=1}^{m} F(\eta_s) \Delta y_s = \int_c^d F(y) dy.$$

$$\therefore \sum_{s=1}^{m} F(\eta_s) \Delta y_s = \int_c^d F(y) dy + \epsilon$$

Hence (ii) becomes

$$\int_a^b F(y) dy + \epsilon + \lim_{m\to\infty} \sum_{s=1}^{m} \epsilon_s \Delta y_s$$

As $m \to \infty$ we get this expression tend to $\int_c^d F(y) dy$ since $\epsilon \to 0$ and ϵ_s can be made less than ϵ_1 which tends to zero and hence $\sum_{s=1}^{m} \epsilon_s \Delta y_s < \epsilon_1 \sum_{s=1}^{m} \Delta y_s$ which tends to zero

~~hence $\sum_{s=1}^{m} e_s \Delta y_s < \epsilon$,~~

$$F(\eta_s) = \int_{f_1(\eta_s)}^{f_2(\eta_s)} f(x, \eta_s)\,dx = \int_{f_1(y)}^{f_2(y)} f(x,y)\,dx$$

Considering y as a constant during integration.

Hence as $n \to \infty$, $m \to \infty$ the sum (C_1) becomes

$$\int_{c}^{d} \int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$

The double integral is therefore evaluted by Considering $f(x,y)$ as a function of x alone but regarding y as a Constant and integrating it between $x = f_1(y)$ and $x = f_2(y)$ and then integrating the resulting function of y between $y = c$ and $y = d$.

Similarly by taking the sum of the terms in each column and then adding these sums

$$\int_R f(x,y)\,dA = \int_{a}^{b} \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

Hence $f(x,y)$ is first Considered as a function of y alone and integrated between $\phi_1(x)$ and $\phi_2(x)$ where the equations of the curves PLQ and PMQ are respectively $y = \phi_1(x)$ and $y = \phi_2(x)$ and then the resulting function of x integrated between $x = a$ and $x = b$

<u>Cor</u>: if the region of integration is a rectangle between the lines $x = a$, $x = b$, $y = c$, $y = d$, then.

$$\int_R f(x,y)\,dA = \int_{a}^{b}\int_{c}^{d} f(x,y)\,dy\,dx = \int_{c}^{d}\int_{a}^{b} f(x,y)\,dx\,dy$$

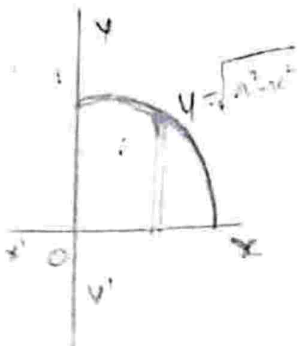
Thus for constant limits the order of integration is immaterial

Note: The integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$ is the integral over the region bounded by the two curves $y=f_1(x)$ and $y=f_2(x)$ for the value of x between a and b. For changing its order one should sketch the region of integration. From the sketch the limits of x and y should be determined as usual.

Examples:

Ex 1: Evalute $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$.

x as constant and y varies from 0 to $\sqrt{a^2-x^2}$.



$$\iint xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a \left[x \cdot \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{x(a^2-x^2)}{2}\, dx\right] = \frac{a^4}{8}$$

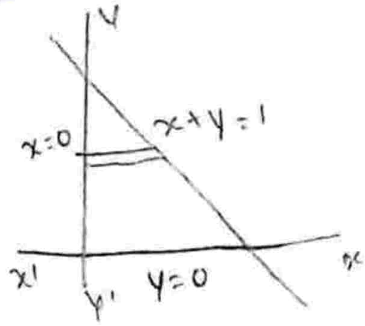
Ex: 2

Evalute $\iint (x^2+y^2)\,dy\,dx$ over the region for which x, y are each ≥ 0 and $x+y \leq 1$

The region is the triangle formed by the lines

$x=0,\ y=0,\ x+y=1$

$\iint (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$



$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$

$= \int_0^1 \left\{x^2(1-x) + \frac{(1-x)^3}{3}\right\} dx$

$= 1/6$

Ex : 3 Change the order of integration in the integral $\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$ and evalute it.

y varies from x^2/a to $2a-x$, y lies between the curves $y = x^2/a$ and $y = 2a-x$, x varies from 0 to a. Hence the region of integration is OPQ. In changing the order of integration, we integrate first with respect to x keeping y constant with elementary strips parallel to x axis. in covering the same region as above the end of these strips exte to the line $x+y=2a$ and to the curve $y=\frac{x^2}{a}$. Hence we divide the region into two parts by the line $y=a$. which passes through P.

Hence for one region x varies from 0 to and for the other region x varies from 0 to In the first region y varies from 0 to $2a-y$ second region y varies from 0 to a and for th varies from a to 2a.

Hence $\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dxd$

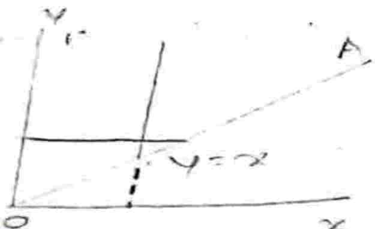
$$= \int_0^a \left[\frac{yx^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} \left[\frac{x^2 y}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a \frac{ay^2}{2} dy + \int_0^{2a} \frac{y(2a-y)^2}{2} dy$$

$$= \frac{3a^4}{8}$$

Ex 4: By changing the order of integration, evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx\, dy$

Let $I = \int_0^\infty dx \int_x^\infty \frac{e^{-y}}{y} dx$



integrate w.r.t y from x to ∞ and then w.r.t x from 0 to ∞

Let OA be the straight line OA + $y = x$

Region of integration is R above OA

Reverse the order of integration

Keep y constant & x varies from 0 to y

Then allow y to vary from 0 to ∞ to cover R.

Hence $I = \int_0^\infty e^{-y} \frac{dy}{y} \int_0^y dx = \int_0^\infty \frac{e^{-y}}{y} dy (x)_0^y$

$$= \int_0^\infty e^{-y} dy = (-e^{-y})_0^\infty = -0 + 1 = 1$$

Double Integral in Polar co-ordinates

In case we use polar co-ordinates we can subdivided the region R in the following way.

Draw the tangents to R which pass through the origin and also draw the circular arcs which are tangent to R with their centres at the origin.

Then divide the radial interval (a, b) into m parts by drawing concentric circular arcs at intervals of Δr_i between $r=a$ and $r=b$ and divide the angular interval (a, b) into n parts by drawing radial lines at intervals of $\Delta\theta_j$ between $\theta=\alpha$ and $\theta=\beta$. By this process we have divided R into infinitesimal "CURVILINEAR" rectangles and partial rectangles the total area of the latter being very small if Δr_i and $\Delta\theta_i$ are sufficiently small.

$$\Delta A_{ij} = \frac{1}{2}(r_i+\Delta r_i)^2\Delta\theta_j - \frac{1}{2}r_i^2\Delta\theta_j$$

$$= \left(r_i+\frac{1}{2}\Delta r_i\right)\Delta r_i \cdot \Delta\theta_j$$

Select a point P_{ij} in each ΔA_{ij} evalute the function $f(r,\theta)$ at each of these points and then from the sum of the products of these functional value and corresponding values of ΔA_{ij}. For convenience chose the r co-ordinates of P_{ij} to be $\xi_i = r_i + \frac{1}{2}\Delta r_i$

Let the θ coordinates of P_{ij} be η_j

Then $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \sum_{ij} f(\xi_i,\eta_j)\,\xi_i\,\Delta r_i\,\Delta\theta_j$ is defined as

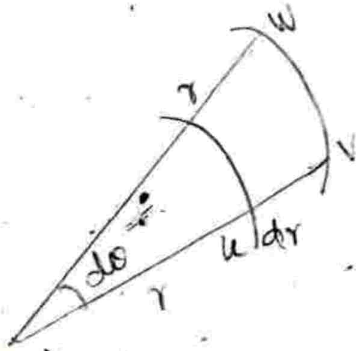
$$\iint_R f(r,\theta)\,r\,dr\,d\theta$$

To evalute the double integral in polar coordin
ates first integrate $f(r,\theta)$ w.r.to r keeping θ Constant
between the limits $r=f_1(\theta)$ and $r=f_2(\theta)$ and
integrate the remaining expression w.r.t θ
between $\theta=\alpha$ and $\theta=\beta$

$$\iint_R f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \left[\int_{r=f_1(\theta)}^{r=f_2(\theta)} r f(r,\theta)\, dr \right] d\theta$$

The adjoint figure may prove helpful as a device for recalling the appropriate expression for dA to be used in setting up repeated integrals in polar Co-ordinates.

Regarding the element dA as a rectangle its area will be product of a pair of adjacent sides say TU and UV



$$dA = (r\,d\theta)\, dr = r\, dr\, d\theta$$

Hence the double integral in cartesian form $\iint_R f(x,y)\, dx\, dy$ transforms into $\iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.
In the latter case we have to use corresponding equation of R in Polar Co-ordinates.

Examples

Ex 1: Evalute $\iint r\sqrt{a^2-r^2}\, dr\, d\theta$ over the upper half of the circle $r=a\cos\theta$

The required integral

$$= \int_0^{\pi/2} \int_{r=0}^{r=a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$$

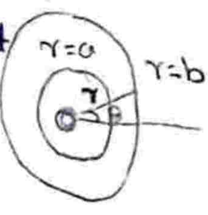
$$= \int_0 \left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} \left\{ (a^2 - a^2\cos^2\theta)^{3/2} - (a^2)^{3/2} \right\} d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2} (1-\sin^3\theta)\,d\theta = \frac{a^3(3\pi-4)}{18}$$

Ex 2: By transforming into polar Co ordinates evalute $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$

By transforming into polar Co-ordinates the two Circles become $r=a$ and $r=b$



$$\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \iint_R \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= \iint_R r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_a^b \cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{b^4-a^4}{4}\int_0^{2\pi} \cos^2\theta\sin^2\theta\,d\theta = \pi\,\frac{b^4-a^4}{16}$$

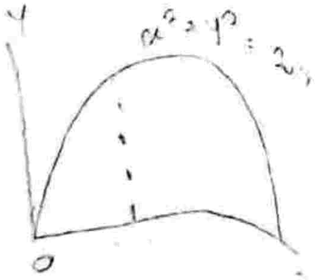
Ex 3: By changing into polar Co-ordinates evalute the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$

The region of integration is semi-circle $x^2+y^2=2ax$ above the x axis. Changing into polars the region becomes $r=2a\cos\theta$ from $\theta=0$ to $\theta=\pi/2$

Hence the required integral

$$= \int_0^{\pi/2} \int_0^{2a\cos\theta} (r^2\cos^2\theta + r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2a\cos\theta} r^3\, dr\, d\theta$$



$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi a^4}{4}$$

TRIPLE INTEGRAL

The triple integral is defined in a manner similar to that of the double integral if $f(x,y,z)$ is continuous and a single valued function of x, y and z over the region of space R enclosed by the surface S. Let R be subdivided into subregions ΔR_{rst} the triple integral of $f(x,y,z)$ over R is defined by

$$\int f(x,y,z)\, dV = \lim_{\substack{n\to\infty,\ m\to\infty,\ p\to\infty}} \sum_{r=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{p} f(\xi_{rst}, \eta_{rst}, \zeta_{rst})\, \Delta V_{rst}$$

In order to evalute the triple integral R is considered to be subdivided by planes parallel to the three coordinates planes.

Then $\Delta V_{rst} = \Delta x_r \Delta y_s \Delta z_t$

By suitably arranging the terms of the sum it can be shown that

$$\int_R f(x,y,z)\,dV = \int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

The limits $z_1, z_2, f_1(z), f_2(z), \varphi_1(y,z), \varphi_2(y,z)$ can be determined from the eqn of the surface S.

NOTE 1:

The student should note that the order of integration is denoted by

$$\int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{z_1}^{z_2} \left[\int_{f_1(z)}^{f_2(z)} \left[\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,dx \right] dy \right] dz$$

The operation of integration being carried out in the rectangles shown in turn starting with the rectangle and working outwards to the outermost rectangle.

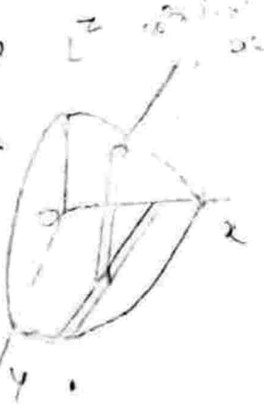
Note 2:

When integrating w.r.t x in the above integra x, y, and z threated as constants and also when integrating w.r.t yz is treated as a constant.

Note 3:

When the integral is given $\iiint f(x,y,z)\,dx\,dy\,dz$ with limit it is often these limits that show the order of integration. if the limits are not constants the integratic should be in the order in which dx, dy, dz is given in the integral.

Example: Evaluate $\iiint xyz \, dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$



To cover the whole positive octant of the sphere $x^2+y^2+z^2=a^2$ z varies from 0 to $\sqrt{a^2-x^2-y^2}$. y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.

The required integral is.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz\, dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{xy(a^2-x^2-y^2)}{2} dy\, dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (xya^2 - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{xy^2a^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= a^6/48$$

CHANGE OF VARIABLES

In certain cases by changing the given variables x, y, z to new variables denoted by u, v, w given by the relations $x = f(u,v,w)$, $y = \varphi(u,v,w)$, $z = \chi(u,v,w)$ evaluation of multiple integrals becomes easy we shall give below first the change of variables in the case of two variables and then extend that to three variables.

JACOBIAN

If $u = f(x,y)$, $v = \varphi(x,y)$ be two continuous functions of the independent variables x and y such that $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y then

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v w.r.t x, y and is denoted by

$$J\left(\frac{u, v}{x, y}\right) \text{ or } \frac{\partial(u,v)}{\partial(x,y)}$$

In the case of three variables u, v, w which are functions of x, y, z the Jacobian of u, v, w w.r.t x, y, z is denoted as

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

and denoted by $J\left(\frac{u, v, w}{x, y, z}\right)$ or $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

TWO IMPORTANT RESULTS REGARDING JACOBIANS

(1) If u, v are functions of x, y and x, y are themselves functions of ξ, η

then $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial(u,v)}{\partial(\xi,\eta)}$

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi} & \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} \\ \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \xi} & \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \eta} \end{vmatrix} \rightarrow (1)$$

but Since $u = f(x,y)$, $v = \varphi(x,y)$

$x = f_1(\xi,\eta)$, $y = \varphi_1(\xi,\eta)$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta}$$

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \xi}$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \eta}$$

Hence (1) $\Rightarrow$ $\begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{vmatrix} = \frac{\partial(u,v)}{\partial(\xi,\eta)}$

---

(2) $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

we have

$$\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(u,v)}$$

but $\frac{\partial(u,v)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix}$

Since u and v are independant Variable

$$\frac{\partial u}{\partial y} = 0 , \frac{\partial v}{\partial u} = 0$$

$$\therefore \frac{\partial(u,v)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = 1$

Cor: we can extend these results to three variables

i) $\frac{\partial(u,v,w)}{\partial(x,y,z)} \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} = \frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)}$

ii) $\frac{\partial(u,v,w)}{\partial(x,y,z)} \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$

Change of variable in the case of two variables

Suppose we have to change the variable in the integral $\int_R \int F(x,y)\, dx\, dy$

to u, v by means of the relation $x = f(u,v)$ & $y = \varphi(u,v)$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \rightarrow (1)$$

$$\partial y = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \rightarrow (2)$$

For integrating $\int_R \int F(x,y)\, dx\, dy$ we first consider y as constant and integrate w.r.t x

When y is considered as constant dy = 0

From (2)

$$0 = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$dv = -\frac{\frac{\partial y}{\partial u} du}{\frac{\partial y}{\partial v}}$$

Substituting this value of dv in (1) we've

$$dx = \frac{\partial x}{\partial u} du - \frac{\frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u} du\right)}{\partial y / \partial v}$$

$$= \frac{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}{\partial y/\partial v} du$$

$$= \frac{\frac{\partial(x,y)}{\partial(u,v)}}{\partial y/\partial v} du$$

$$\int_R \int F(x,y)\, dx\, dy = \int dy \int F_1(y,u) \frac{\frac{\partial(x,y)}{\partial(u,y)}}{\partial y/\partial v} du$$

Changing the order of integration the integral becomes

$$\int du \int F_1(y,v) \frac{\frac{\partial(x,y)}{\partial(u,y)}}{\partial y/\partial v} dy$$

$$x = f(u,v), \quad y = \varphi(u,v)$$

Eliminating v we get a relation between u, x and y and so x can be expressed in terms of u and y

Hence F(x,y) well transform into $F_1(u,y)$

To integrate (3), u is considered constant and so $du = 0$

From equating (2) we get $dy = \frac{\partial y}{\partial v} dv$. Equating (3) becomes than

$$\int du \int \frac{F_1\{\varphi(u,v), u\}}{\partial y/\partial v} \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial y}{\partial v} dv$$

$$\iint F_2(u,v) \frac{\partial(x,y)}{\partial(u,v)} du\, dv$$

Here $F_2(u,v)$ is the result of substituting $f(u,v)$ and $\varphi(u,v)$ for x, y in $F(x,y)$ and it is understood that the limits are now those appropriate to new variables.

Alliter:

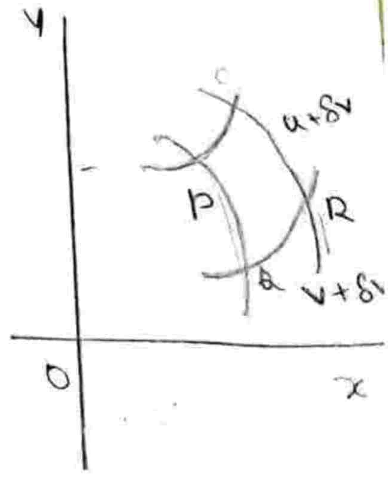
x, y being functions of u and v, u and v are function of x and y. Hence if a particular value of u determines a curve PS, a neighbouring value $u+\delta u$ will determine a neighbouring curve QR. Similarly v and $v+\delta v$ will determine two curve PQ and RS

Let the co-ordinates of P be (x,y)

At Q, $\delta v = 0$

$$dx = \frac{\partial x}{\partial u} \delta u$$

$$dy = \frac{\partial y}{\partial u} \delta u$$



Hence the Coordinates of Q are

$$\left(x + \frac{\partial x}{\partial u} \delta u, \; y + \frac{\partial y}{\partial u} \delta u\right)$$

Similarly the Coordinates of R and S are respectively

$$\left(x + \frac{\partial x}{\partial u} \delta u + \frac{\partial x}{\partial v} \delta v, \; y + \frac{\partial y}{\partial u} \delta u + \frac{\partial y}{\partial v} \delta v\right)$$

and $$\left(x + \frac{\partial x}{\partial v} \delta v, \; y + \frac{\partial y}{\partial v} \delta v\right)$$

Slope of PQ is $\dfrac{\frac{\partial y}{\partial u}\delta u}{\frac{\partial x}{\partial u}\delta u}$ and Slope of RS is $\dfrac{\frac{\partial y}{\partial u}\delta u}{\frac{\partial y}{\partial u}\delta u}$

Hence PQ is Parallel to RS. Similarly PS is Parallel to QR

$\therefore$ PQRS is Parallelogram

Hence we can regard the area enclosed by the four curves as a parallelogram

$$\text{Area of PQRS} = \begin{vmatrix} x & y & 1 \\ x+\frac{\partial x}{\partial u}\delta u & y+\frac{\partial y}{\partial u}\delta u & 1 \\ x+\frac{\partial x}{\partial v}\delta v & y+\frac{\partial y}{\partial v}\delta v & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ \frac{\partial x}{\partial u}\delta u & \frac{\partial y}{\partial u}\delta u & 0 \\ \frac{\partial x}{\partial v}\delta v & \frac{\partial y}{\partial v}\delta v & 0 \end{vmatrix} \quad \begin{matrix} R_2-R_1 \\ R_3-R_1 \end{matrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \delta u\, \delta v$$

$$= \frac{\partial(x,y)}{\partial(u,v)}\, \delta u\, \delta v.$$

Hence the rule is Change $dx\,dy$ into $\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$.

Change of variables in the case of three variables.

This argument can be extended without difficulty to triple integrals with a Similar result, viz:

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint F(u,v,w)\,\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$$

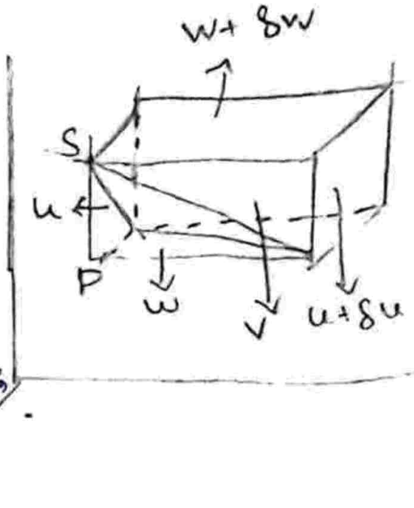
Where x, y, z are given in terms of u, v and w and limits are changed to suit the new variables.

Aliter:-

Let $x = f(u,v,w)$

$y = \varphi(u,v,w)$

$z = \varphi(u,v,w)$



Let P, Q, R, S the points determined by the parameters u, v, w (P); $u+\delta u$, v, w (Q); u, $v+\delta v$, w (R) and u, v, $w+\delta w$ (S)

$$dx = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v + \frac{\partial x}{\partial w}\delta w$$

Hence the point Q is

$$x + \frac{\partial x}{\partial u}\delta u,\ y + \frac{\partial y}{\partial u}\delta u,\ z + \frac{\partial z}{\partial u}\delta u$$

R is $x + \frac{\partial x}{\partial v}\delta v,\ y + \frac{\partial y}{\partial v}\delta v,\ z + \frac{\partial z}{\partial v}\delta v$

and S is $x + \frac{\partial x}{\partial w}\delta w,\ y + \frac{\partial y}{\partial w}\delta w,\ z + \frac{\partial z}{\partial w}\delta w$

Hence Vol. PQRS

$$= \frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ x+\frac{\partial x}{\partial u}\delta u & y+\frac{\partial y}{\partial u}\delta u & z+\frac{\partial z}{\partial u}\delta u & 1 \\ x+\frac{\partial x}{\partial v}\delta v & y+\frac{\partial y}{\partial v}\delta v & z+\frac{\partial z}{\partial v}\delta v & 1 \\ x+\frac{\partial x}{\partial w}\delta w & y+\frac{\partial y}{\partial w}\delta w & z+\frac{\partial z}{\partial w}\delta w & 1 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ \frac{\partial x}{\partial u}\delta u & \frac{\partial y}{\partial u}\delta u & \frac{\partial z}{\partial u}\delta u & 0 \\ \frac{\partial x}{\partial v}\delta v & \frac{\partial y}{\partial v}\delta v & \frac{\partial z}{\partial v}\delta v & 0 \\ \frac{\partial x}{\partial w}\delta w & \frac{\partial y}{\partial w}\delta w & \frac{\partial z}{\partial w}\delta w & 0 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$= -\frac{1}{6}\,\delta u\,\delta v\,\delta w \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= -1/6\,\delta u\,\delta v\,\delta w\,\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Element of volume $dx\,dy\,dz$

= a parallelopiped whose volume is Six times the volume of PQRS

$$= \frac{\partial(x,y,z)}{\partial(u,v,w)}\,\delta u\,\delta v\,\delta w$$

Hence the rule change $dx\,dy\,dz$ into $\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw$

Transformation from Cartesian to polar co-ordinates:

Let the polar coordinates of point p whose Cartesian co-ordinates (x,y) be (r,θ)

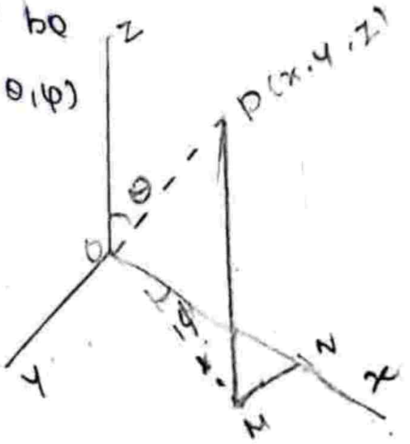
Then $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence $dx\,dy$ has to be changed to $r\,dr\,d\theta$

Transformation from Cartesian to spherical polar co-ordinates

Let the coordinates of P be (x, y, z) in Cartesian and (r, θ, φ) in spherical co-ordinates



$$z = MP = r\cos\theta$$
$$OM = r\sin\theta$$

$$ON = OM\cos\varphi = r\sin\theta\cos\varphi$$
$$NM = OM\sin\varphi = r\sin\theta\sin\varphi$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$\frac{\partial(x, y, z)}{\partial(r, \varphi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\varphi\sin\theta & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix}$$

$$= -r^2\sin\theta.$$

Hence the $dx\,dy\,dz$ will change into $-r^2\sin\theta\,dr\,d\varphi\,d\theta$

Examples:

Ex 1: Given that $x+y=u$, $y=uv$, change the variables to u, v in the integral $\iint [xy(1-x-y)]^{1/2}\,dx\,dy$ taken over the area of the triangle with sides $x=0$, $y=0$, $x+y=1$ and evalute it.

$$x+y=u \qquad y=uv$$

$$x=u(1-v), \quad y=uv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

The area of triangle with sides $x=0, y=0$, $x+y=1$ transforms into the area of the Square with Sides

$$u=0, u=1, v=0, v=1$$

$$\iint [xy(1-x-y)]^{1/2} dx dy$$

$$= \iint [u^2 v(1-v)(1-u)]^{1/2} u \, du dv$$

$$= \int_0^1 \int_0^1 u^2 (1-u)^{1/2} v^{1/2} (1-v)^{1/2} du dv$$

$$= \int_0^1 u^2 (1-u)^{1/2} du \int_0^1 v^{1/2} (1-v)^{1/2} dv$$

$$= \frac{2\pi}{105} \text{ on Simplification.}$$

Ex 2: Evalute $\iint_R (x-y)^4 e^{x+y} dx dy$ where R is the square with verticles $(1,0), (2,1), (1,2)$ and $(0,1)$

The sides of the square are

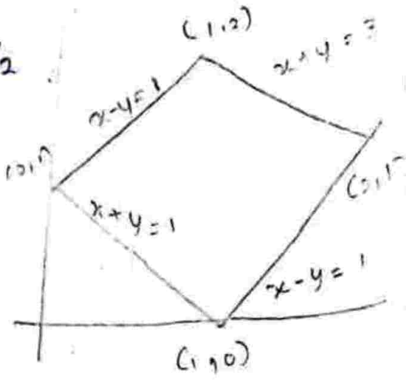
$$x+y=1 \quad , x+y=3$$

$$x-y=1 \quad , x-y=-1$$

Transform the variables x and y to u and v Such that

$$x+y=u, \quad x-y=v$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -2, \quad \therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = -\tfrac{1}{2}.$$



Hence $\iint (x-y)^4 e^{x+y}\, dx\,dy =$

$$= \int_{v=1}^{-1}\int_{u=1}^{3} v^4 e^u \left(-\tfrac{1}{2}\right)$$

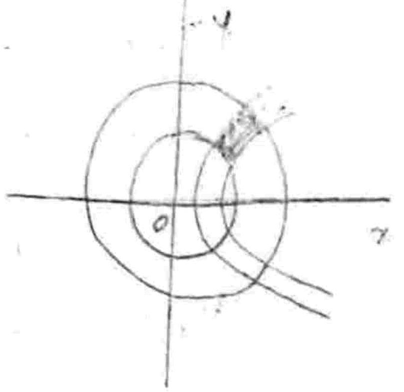
$$= -\tfrac{1}{2}\int_1^{-1} v^4\,dv \int_1^3 e^u\,du$$

$$= -\frac{1}{2}\left[\frac{v^5}{5}\right]_1^{-1}\left[e^u\right]_1^3 = \frac{e^3-e}{5}$$

Ex: 3 : Evaluate $\iint_R xy\,dx\,dy$ **in the first quadrant where R is the region bounded by the hyperbolas** $x^2-y^2=a^2$ **and** $x^2-y^2=b^2$ **and the circles** $x^2+y^2=c^2$ **and** $x^2+y^2=d^2$ $(0<a<b<c<d)$.

Transform the variables x and y to u and v such that

$x^2-y^2=u$ and $x^2+y^2=v$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix} = 8xy$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{8xy}$$

$$\iint xy\,dx\,dy = \iint xy\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \iint xy\,\frac{1}{8xy}\,du\,dv$$

$$= \frac{1}{8}\iint du\,dv$$

In this transformation the Region R becomes the ~~rectangle~~ rectangle between $u=a^2$, $u=b^2$, $v=c^2$, $v=d^2$

Hence the integral is

$\frac{1}{8}$ area of the rectangle $= \frac{1}{8}(b^2-a^2)(d^2-c^2)$

Ex 4: Find the area of the curvilinear quadrilateral bounded by the four parabolas.

$$y^2 = ax,\ y^2 = bx,\ x^2 = cy,\ x^2 = dy$$

Putting $y^2 = u^3 x$ and $x^2 = y v^3$, u varies from $a^{1/3}$ to $b^{1/3}$ while v varies from $c^{1/3}$ to $d^{1/3}$.

Solving $x = uv^2$ and $y = u^2 v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v^2 & 2uv \\ 2uv & u^2 \end{vmatrix} =$$

$$= -3u^2v^2$$

Hence the area required $= \iint_R dx\,dy$

$$= \int_{v=c^{1/3}}^{v=d^{1/3}} \int_{u=a^{1/3}}^{u=b^{1/3}} 3u^2v^2\,du\,dv = \frac{(b-a)(d-c)}{3}$$

Ex 5: Prove that $\iint_D e^{-x^2-y^2}\,dx\,dy = \frac{1}{4}\pi(1-e^{-R^2})$ where D is the region $x \geq 0$, $y \geq 0$ and $x^2+y^2 \leq R^2$

Changing to polar co ordinates, the integral becomes $\iint_D -e^{-r^2}\,r\,dr\,d\theta$ on simplification

Where r varies from 0 to R and θ varies from 0 to $\pi/2$

$$= \int_0^{\pi/2} \theta \int_0^R +e^{-r^2}\,r\,dr = \frac{\pi}{2}\left[\frac{-e^{-r^2}}{2}\right]_0^R$$

$$= \frac{\pi}{2}(1-e^{-R^2})$$

Ex 6: Evalute $\iint xyz\, dx\, dy\, dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical co-ordinates.

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$dx\, dy\, dz = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\, dr\, d\theta\, d\varphi$$

$$= -r^2\sin\theta\, dr\, d\theta\, d\varphi$$

To cover the positive octant of the sphere r varies from 0 to a, φ varies from 0 to $\pi/2$ and θ varies from $\pi/2$ to 0

Required integral

$$= \int_{r=0}^{a}\int_{\varphi=0}^{\pi/2}\int_{\theta=\pi/2}^{0} r^3 \sin^2\theta\cos\theta \sin\varphi\cos\varphi\,(-r^2\sin\theta)\, dr\, d\theta\, d\varphi$$

$$= \int_{r=0}^{a}\int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^5 \sin\varphi\cos\varphi\sin^3\theta\cos\theta\, dr\, d\theta\, d\varphi$$

$$= \int_0^a r^5\, dr \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi \int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta$$

$$= \frac{a^6}{48} \text{ on Simplification}$$

Ex 7: Use the Substitution $x+y+z=u$, $y+z=uv$, $z=uvw$ to evaluate the integral $\iiint [xyz(1-x-y-z)]^{1/2}\, dx\, dy\, dz$ taken over the tetrahedral volume enclosed by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

Here $x = u(1-v)$

$y = uv(1-w)$

$z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = u^2v$$

The tetrahedral volume is covered by taking the limits for u from 0 to 1, v from 0 to 1 and w from 0 to 1.

∴ Required integral

$$= \int_0^1 \int_0^1 \int_0^1 \left[u^3 v^2 w(1-u)(1-v)(1-w)\right]^{1/2} u^2 v \, du \, dv \, dw$$

$$= \int_0^1 u^{1/2}(1-u)^{1/2} du \int_0^1 v^2 (1-v)^{1/2} dv \int_0^1 w^{1/2}(1-w)^{1/2} dw$$

Putting $u = \sin^2\theta$, $v = \sin^2\varphi$, $w = \sin^2 t$, the integral becomes

$$\int_0^{\pi/2} 2\sin^8\theta\cos^2\theta \, d\theta \int_0^{\pi/2} 2\sin^5\varphi\cos\varphi \, d\varphi \int_0^{\pi/2} 2\sin^2 t\cos^2 t \, dt$$

$= \pi^2/1920$ on simplification.

UNIT-5

BETA AND GAMMA FUNCTION

Definition:-

1. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for $m>0, n>0$, is known as Beta function and is denoted by $\beta(m,n)$

2. $\int_0^\infty x^{n-1}e^{-x}\,dx$ for $n>0$ is known as Gamma function and is denoted by $\Gamma(n)$

Corollary:-1

$$\Gamma(n+1) = n!$$

corollary:-2

$$\Gamma(n+a) = (n+a-1)(n+a-2)\cdots a\Gamma(a)$$

properties of Beta function

(1) $\beta(m,n) = \beta(n,m)$

$\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

putting $x=1-y$ we have

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}[1-(1-y)]^{n-1}(-dy)$$

$$= \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

11) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \, dx$.

Relation b/w Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Corallary :- 1

$\Gamma(1/2) = \sqrt{\pi}$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

put $m = n = 1/2$

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$$

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2}$$

$$= 2(\pi/2 - 0)$$

$$\frac{\Gamma(1/2)\Gamma(1/2)}{1} = \pi$$

$$[\Gamma(1/2)]^2 = \pi$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Problem:- 1

Evaluate $\int_0^1 x^m (\log 1/x)^n dx$

put $\log 1/x = t$

$1/x = e^t$

$x^{-1} = e^{t}$

$dx = -e^{-t} dt$

$= e^{-mt-t}$

$= e^{-(m+1)t}$

$$\int_0^1 x^m (\log 1/x)^n dx = \int_\infty^0 e^{-mt} (t^n)(-e^{-t}) dt$$

$$= \int_0^\infty e^{-(m+1)t} t^n dt$$

put

$(m+1)t = y$

$dt = \frac{1}{m+1} dy$

Then the integral on this substitution becomes

$$= \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^n \left(\frac{1}{m+1}\right) dy$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n dy$$

$$= \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

Problem:- 2

$\int_0^\infty e^{-x^2} dx$

put $x^2 = t$, $x = \sqrt{t}$

$2x dx = dt$

$dx = dt/2x$

$= dt/2\sqrt{t}$

$$\int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}} dt$$

$= \frac{1}{2}\int_0^\infty t^{-1/2} e^{-t}\,dt$

$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{1/2-1}\,dt$

$= \frac{1}{2}\Gamma(1/2)$ $\quad \Gamma(1/2) = \sqrt{\pi}$

$= \sqrt{\pi}/2$

problem:- 3

Evaluate (i) $\int_0^1 x^7 (1-x)^8\,dx$

$\left[\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right.$

$\int_0^1 x^7(1-x)^8\,dx = \int_0^1 x^{8-1}(1-x)^{9-1}\,dx$

$= \beta(8,9)$ $\quad \Gamma(n+2) = n+1$

$= \frac{\Gamma(8)\Gamma(9)}{\Gamma(17)}$ $\quad \Gamma(n+1) = n!$

$\Gamma(n) = n-1!$

$= \frac{7!\,8!}{16!}$

problem:- 4

$\int_0^{\pi/2} \sin^7\theta \cos^5\theta\, d\theta.$

$\left[\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n) \right]$

$\int_0^{\pi/2} \sin^7\theta\cos^5\theta\, d\theta = \int_0^{\pi/2} \sin^{2(4)-1}\theta \cos^{2(3)-1}\theta\, d\theta$

$= \frac{1}{2}\beta(4,3)$

$= \frac{1}{2}\frac{\Gamma 4\,\Gamma 3}{\Gamma 7}$

$= \frac{1}{2} \frac{2!\,2!}{6!}$

$= \frac{2!}{6!}$

$= \frac{2!}{6 \times 5 \times 4 \times 3 \times 2!}$

$= \frac{1}{180}$

problem:- 5

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta$$

$$\left[\int_0^{\pi/2} \sin^{p-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)}\right]$$

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \int_0^{\pi/2} \sin^{11-1}\theta \, d\theta$$

$$= \frac{1}{2} \frac{\Gamma(11/2)\Gamma(1/2)}{\Gamma(11/2 + 1/2)}$$

$$= \frac{1}{2} \frac{\Gamma(10/2 + 1/2)\Gamma(1/2)}{\Gamma(6)}$$

$$\Gamma(n+a) = (n+a-1)(n+a-2) \ldots a\Gamma(a)$$

$$= (11/2 - 1)(11/2 - 2)(11/2 - 3)(11/2 - 4)(11/2 - 5)$$

$$= 9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma(1/2)$$

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{1}{2} \frac{9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma(1/2) \cdot \Gamma(1/2)}{5!}$$

$$= \frac{9 \cdot 7 \cdot 5 \cdot 3 [\Gamma(1/2)]^2}{5! \, 2^6} = \frac{63(\sqrt{\pi})^2}{512}$$

$$= \frac{63\pi}{512}$$

problem:- 6

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta = \int_0^{\pi/2} \sin^{2(7/2)-1}\theta \cos^{2(4)-1}\theta \, d\theta$$

$$= \frac{1}{2} \beta(7/2, 4)$$

$$= \frac{1}{2} \frac{\Gamma(7/2)\Gamma(4)}{\Gamma(15/2)}$$

$$= \frac{\frac{1}{2}\Gamma(6/2 + 1/2)\, 3!}{\Gamma(14/2 + 1/2)}$$

$$= \frac{1}{2} \frac{5/2 \cdot 3/2 \cdot 1/2\, \Gamma(1/2) \cdot 3!}{13/2 \cdot 11/2 \cdot 9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2\, \Gamma 1/2}$$

$$= \frac{1}{2} \frac{1 \times 2 \times 3 \times 2^4}{13 \times 11 \times 9 \times 7}$$

$$= \frac{3\times 16}{13\cdot 11\cdot 9\cdot 7}$$

$$= \frac{16}{13\cdot 11\cdot 7\cdot 3}$$

Problem:- 7

$$\int_0^{\pi/2} \sin^4\theta \cos^6\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^4\theta \cos^6\theta \, d\theta = \int_0^{\pi/2} \sin^{2(5/2)-1}\theta \cos^{2(7/2)-1}\theta \, d\theta$$

$$= \frac{1}{2} B(5/2, 7/2)$$

$$= \frac{1}{2} \frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(12/2)}$$

$$= \frac{\frac{1}{2}\Gamma(5/2)\Gamma(7/2)}{\Gamma(6)}$$

$$= \frac{\frac{1}{2}\Gamma(4/2+1/2)\Gamma(6/2+1/2)}{5!}$$

$$= \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{5!}$$

$$= \frac{3\cdot 5\cdot 3(\sqrt{\pi})^2}{1\times 2\times 3\times 4\times 5\times 2^6}$$

$$= \frac{3\pi}{8\times 64}$$

$$= \frac{3\pi}{512}$$

problem:- 8

$$\int_0^{\pi/2} \sin^7\theta \cos^7\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^7\theta \cos^7\theta \, d\theta = \int_0^{\pi/2} \sin^{2(4)-1}\theta \cos^{2(4)-1}\theta \, d\theta$$

$$= \frac{1}{2}\beta(4,4)$$

$$= \frac{\frac{1}{2}\Gamma(4)\Gamma(4)}{\Gamma(8)}$$

$$= \frac{1}{2}\frac{3!\,3!}{7!}$$

$$= \frac{1}{2}\frac{2!\cdot 3\cdot 3!}{3!\times 4\times 5\times 6\times 7}$$

$$= \frac{1}{20\times 14}$$

$$= \frac{1}{280}$$

problem:- 9

$$\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta \, d\theta = \frac{1}{2}\beta\left[\frac{p}{2}, \frac{q}{2}\right]$$

$$\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}} \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}} \, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{3/2 - 1}\theta \cos^{1/2 - \frac{1}{2}}\theta \, d\theta$$

$$= \frac{1}{2} B\left[\frac{3}{4}, \frac{1}{4}\right]$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma\left(\frac{3+1}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \Gamma(1 - 1/4)\Gamma(1/4)$$

$$\because \Gamma(1-n)\Gamma n = \frac{\pi}{\sin n\pi}$$

$$= \frac{1}{2} \frac{\pi}{\sin \pi/4}$$

$$= \frac{\pi}{2 \times 1/\sqrt{2}} = \frac{\pi}{\sqrt{2} \times \sqrt{2} \times 1/\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}}$$

Problem:-10

$$\int_0^{\infty} e^{-x^3} dx$$

put $x^3 = t$ $x = t^{1/3}$

$3x^2 dx = dt$

$dx = dt/3x^2$

$= \frac{dt}{3(t^{2/3})}$

$$\int_0^{\infty} e^{-x^3} dx = \int_0^{\infty} e^{-t} \frac{1}{3(t^{2/3})} dx$$

$= \frac{1}{2} \int_0 e^{-t} t^{1/3} dt$ $\quad 1/3 - 1$ $\quad -2/3$

$\frac{1}{2} \int_0^\infty e^{-t} t^{1/3 - 1} dt$

$= \frac{1}{2} \Gamma(1/3)$

problem :- 11

Express $\int_0^1 x^m (1-x^n)^p dx$ Interms of gamma function and evaluate the Integral $\int_0^1 x^5 (1-x^3)^{10} dx$

Put $x^n = y$

$x = y^{1/n}$

$x^m = y^{m/n}$

$n x^{n-1} dx = dy$

$dx = \frac{dy}{n x^{n-1}}$

$dx = \frac{dy}{n x^n x^{-1}}$

$= \frac{dy}{n y \times y^{-1/n}}$

$dx = \frac{dy}{n y^{(\frac{n-1}{n})}}$

$\int_0^1 x^m (1-x^n)^p dx = \int_0^1 y^{m/n} (1-y)^p \frac{dy}{n y^{(\frac{n-1}{n})}}$

$= \frac{1}{n} \int_0^1 y^{m/n - (\frac{n-1}{n})} (1-y)^p dy$

$= 1/n \int_0^1 y^{\frac{m-n+1}{n}} (1-y)^p dy$

$= 1/n \, \beta\left(\frac{m-n+1}{n} + 1, \; p+1\right)$